Dirichlet heat kernels and exit times for subordinate Brownian motions

Michał Ryznar

Wrocław University of Technology

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Based on joint work in progress with K. Bogdan and T.Grzywny

- Let ψ be the Laplace exponent of a subordinator S_t. Assume that ψ is a Bernstein function:

$$\psi(\lambda) = b\lambda + \int_0^\infty (1 - e^{-\lambda x})\mu(dx),$$

where the measure μ: $\int_0^\infty (1 - e^{-x})\mu(dx) < \infty$. In this talk $b = 0$.

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- Examples:
β-stable subordinator, $\psi(\lambda) = \lambda^\beta, 0 < \beta < 1$
geometric stable, $\psi(\lambda) = \ln(1 + \lambda^\beta), 0 < \beta \leq 1$
relativistic stable $\psi(\lambda) = (\lambda + m^{1/\beta})^\beta - m, 0 < \beta < 1, m \geq 0$

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  $\mathcal{A} = -\left(-\Delta + m^{2/\alpha}\right)^{\alpha/2} + m, 0 < \alpha < 2$

- The process X_t has a density of the form

$$p(t,x,y) = p(t,x-y) = \int_0^\infty g(u,x-y)P(S_t \in du), \quad (0.1)$$

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- The stable case $\nu(x) = \frac{A}{|x|^{d+\alpha}}$, but in the geometric stable $\nu(x) \approx \frac{1}{|x|^d}, |x| \leq 1$.

Survival probability. Dirichlet heat kernel.

- The first exit (survival) time from open $D \subset \mathbb{R}^d$:

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- In particular p_D yields the **probability of surviving** time t:

$$P^x(\tau_D > t) = \int p_D(t,x,y)dy.$$

and the **Green function** of D:

$$G_D(x,y) = \int_0^\infty p_D(t,x,y)dt.$$

Exact formulas for the stable case

- The survival probability for a halfspace for $\alpha = 1$ was computed by Darling (1956). Recent progress: Doney (2008), Graczyk and Jakubowski (2009), Doney and Savov (2010), Kuznetsov (2010), Kuznietzov and Halubek (2011)

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- For more complicated sets - even intervals such formulas seem to be out of reach.

Classical Laplacian

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- Explicit qualitatively sharp estimates for the classical heat kernel in $C^{1,1}$ domains, $d \geq 3$: Zhang (2002)

$$p_D(t,x,y) \geq C^{-1}\left(1 \wedge \frac{\delta_D(x)}{t^{1/2}}\right) p(t,cx,cy) \left(1 \wedge \frac{\delta_D(y)}{t^{1/2}}\right),$$

$$p_D(t,x,y) \leq C\left(1 \wedge \frac{\delta_D(x)}{t^{1/2}}\right) p(t,c^{-1}x,c^{-1}y) \left(1 \wedge \frac{\delta_D(y)}{t^{1/2}}\right),$$

for $x, y \in \mathbb{R}^d$, $t < 1, c, C \geq 1$

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  for $x, y \in \mathbb{R}^d$, $t < 1, c, C \geq 1$
- Qualitatively sharp heat kernel estimates for Lipschitz domains: Varopulous (2003)

Estimates for fractional Laplacian

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$$p_D(t,x,y) \approx \left(1 \wedge \frac{\delta_D^{\alpha/2}(x)}{\sqrt{t}}\right) p(t,x,y) \left(1 \wedge \frac{\delta_D^{\alpha/2}(y)}{\sqrt{t}}\right) .$$

Here $\delta_D(x) = \mathrm{dist}(x, D^c)$.

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- Circular cones V: Bogdan and Grzywny (2008)

$$p_V(t,x,y) \approx P^x(\tau_V > t)\, p(t,x,y)\, P^y(\tau_V > t)\,, \quad t > 0.$$

κ-fat domains

Definition

Let $x \in \mathbb{R}^d$, $r > 0$ *and* $0 < \kappa \leq 1$. *We say that open* D *is* (κ, r)*-fat at* x *if there is a ball* $B(A, \kappa r) \subset D \cap B(x, r)$.

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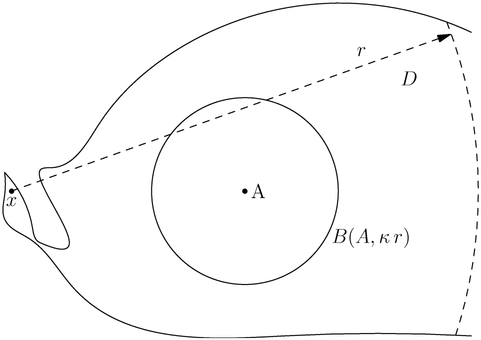

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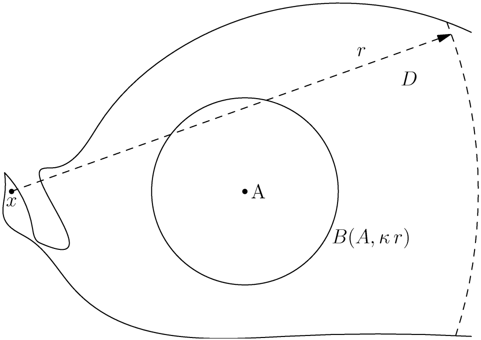

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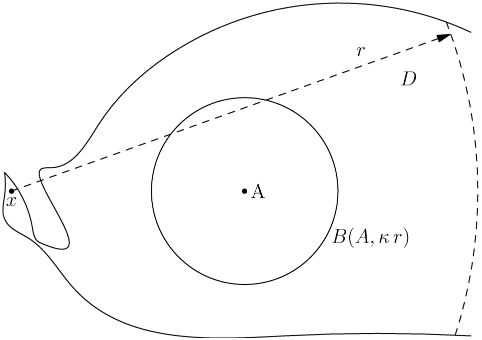

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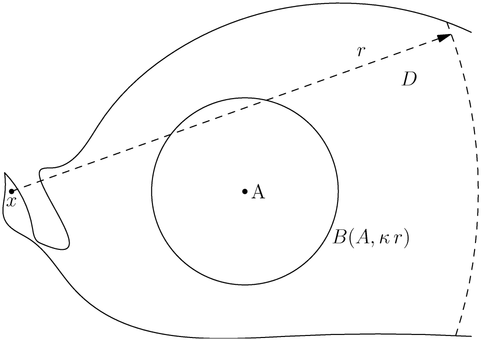

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Let $x \in \mathbb{R}^d$, $r > 0$ *and* $0 < \kappa \le 1$. *We say that open* D *is* (κ, r)*-fat at* x *if there is a ball* $B(A, \kappa r) \subset D \cap B(x, r)$. *If this is true for every* $x \in D$ *then we say that* D *is* (κ, r)*-fat. We say that* D *is* κ*-fat if there is* $R > 0$ *such that* D *is* (κ, r)*-fat for all* $r \in (0, R]$.

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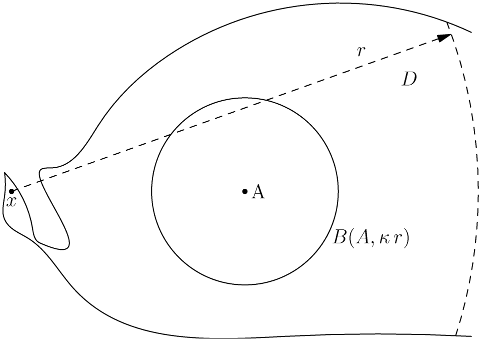
We have:
$\delta_D(A_r) \approx r \vee \delta_D(x)$.



$C^{1,1}$ domains

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Open D *is of class* $C^{1,1}$ *at scale* $r > 0$ *if for every* $Q \in \partial D$ *there exist balls* $B(x', r) \subset D$ *and* $B(x'', r) \subset D^c$ **tangent** *at* Q*.*

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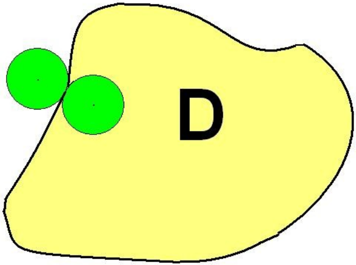

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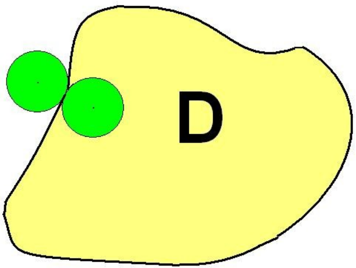
If D is $C^{1,1}$ at scale r then it is $(1/2, p)$-fat for all $p \in (0, r]$.



General two-sided estimate of Dirichlet heat kernel. Stable case.

Theorem (Bogdan, Grzywny and R, 2010)

If D is κ-fat then there is $C = C(\alpha, D)$ such that for all $x, y \in \mathbb{R}^d$,

$$p_D(t,x,y) \overset{C}{\approx} P^x(\tau_D > t)\, p(t,x,y)\, P^y(\tau_D > t)\,, \quad 0 < t \leq 1.$$

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where $C = C(d,\alpha,\kappa)$.

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For $\alpha = d = 1$, $s_D(x) \approx \log(1 + \delta_D^{1/2}(x))$, and so $s_D(A_{t^{1/\alpha}}(x)) = \log(1 + (t \vee \delta_D(x))^{1/2})$,

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Here all $t > 0$ and $x, y \in \mathbb{R}^d$ are allowed.

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In the transient case $d > \alpha$ similar estimates hold (obtained also by Chen and Tokle (2010)).

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- Let B_t be a Brownian motion (with variance $2t$) independent of the subordinator S_t. From now on we consider

$$X_t = B_{S_t}$$

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- **Goal**: find good estimates of **Dirichlet heat kernel**:

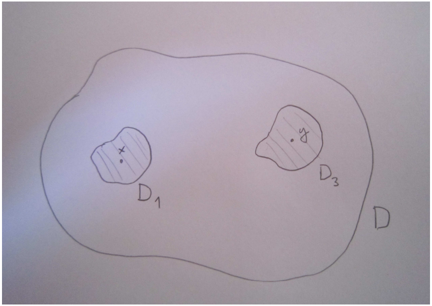
$$p_D(t,x,y) = p(t,x,y) - E^x[\tau_D < t; p(t-\tau_D, X_{\tau_D}, y)],$$

D - open set with a smooth boundary.

Lemma

Consider open $D_1, D_3 \subset D$ *such that* $\operatorname{dist}(D_1, D_3) > 0$*. Let* $D_2 = D \setminus (D_1 \cup D_3)$*. If* $x \in D_1$*,* $y \in D_3$ *and* $t > 0$*, then*

$$\begin{aligned} p_D(t,x,y) &\leq P^x(X_{\tau_{D_1}} \in D_2) \sup_{s<t,\, z\in D_2} p(s,z,y) \\ &+ (t \wedge E^x \tau_{D_1}) \sup_{u\in D_1,\, z\in D_3} \nu(z-u)\,. \end{aligned}$$



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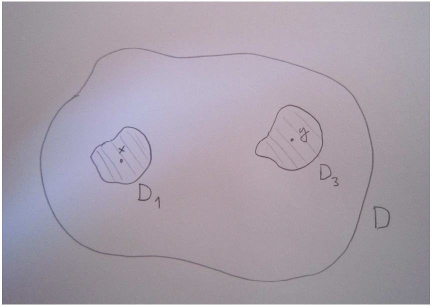
$$p_D(t,x,y) \geq t P^x(\tau_{D_1} > t)\, P^y(\tau_{D_3} > t) \inf_{u \in D_1,\, z \in D_3} \nu(z-u)\,.$$



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- $\{Y_t\}_{t\geq 0}, Y_0 = 0$: Lévy process in $\mathbb{R}$, not a subordinator

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  For our purpose we take Y_t as a one-dimensional projection of SBM X_t. We know that $V(r) \approx \frac{1}{\sqrt{\psi(r^{-2})}}$.

Scaling conditions for the Laplace exponent of the subordinator

LSC condition for ψ. There are $C > 0$ and $0 < \sigma < 1$ and $\theta_0 > 0$ such that

$$\psi(\lambda\theta) \geq C\lambda^{\sigma}\psi(\theta), \qquad \lambda \geq 1, \quad \theta > \theta_0$$

USC condition for ψ. There are $C^* > 0$ and $0 < \sigma^* < 1$ and $\theta_0^* > 0$ such that

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USC condition for ψ'. There is $C > 0$ and $\delta < 0$ and $\theta_0 > 0$ such that

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Estimates of the mean exit time

Lemma

Assume **USC** *for ψ'. For $r \le R$*

$$CV(\delta(x))V(r) \le E^x \tau_{B(0,r)} \le 2V(\delta(x))V(r),$$

where the constant C depends on R and is decreasing in R.

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- The upper bound is true for any rotationally invariant Lévy process.
- Recall that for any subordiante Brownian motion $V(r) \approx \frac{1}{\sqrt{\psi(r^{-2})}}$.

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- The above estimates are based on the estimates of the action of the generator on a test function which is built from V. The idea comes from the paper of Kim, Song and Vondracek (2011).

Lemma

Let $R \leq 1$. There are C_1, C_2 such that for $t \leq C_1 V^2(R)$

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Let $D = B(0,1)^c$ and let $t \leq 1$. There is C such that

$$P^x(\tau_D > t) \leq C \frac{V(\delta_D(x))}{\sqrt{t}}.$$

Good properties for the transition density of the free process

Suppose that for $t \le 1$ and $|y| \le R$:

[A1] If $V(|y|)^2 \le t$, then $C_1 p(t,0) \le p(t,y) \le p(t,0)$.
[A2] $C_2^{-1} p(t,y) \le t\nu(y) \le C_2 p(t,y)$, provided $t \le V(|y|)^2$

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If $t \le V(|y|)^2$ then **A2** implies

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$$p_t(y) \approx p(t,0) \wedge t\nu(y), t \le 1, |y| \le R$$

Theorem (Bogdan, Grzywny, R 2012)

Assume LSC and USC for the subordinator.
(A) Then the conditions **A1** *and* **A2** *hold.*
(B) Suppose that D is a bounded $C^{1,1}$ at scale r_0. There are constants C, c_1, c_2 dependent ψ, r_0 and the diameter of D such that

$$\begin{aligned} & C^{-1} P^x(\tau_D > t)\, p(t, c_1(x-y))\, P^y(\tau_D > t) \\ \leq \; & p_D(t,x,y) \\ \leq \; & C P^x(\tau_D > t)\, p(t, c_2(x-y))\, P^y(\tau_D > t)\,, \quad 0 < t < 1. \end{aligned}$$

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In the recent years there has been a big progress in studying subordinate Brownian motions. For example Chen, Kim, Song in a number of papers (2010-2012) obtained sharp heat kernel estimates in particular cases of subordinate Brownian motions for bounded domains and some unbounded ones.

Lower bound in the case $D = B(0,1)$

Lemma

Suppose that $r < 1$ *and* $R > 0$. *There are* $c_1, c^*, C = C(R)$ *such that for* $t \le c^* V^2(r)$ *we have*

$$p_{B(x,r)\cup B(y,r)}(t,u,v) \ge Cp(t, c_1(x-y)), \quad |x-y| < R.$$

where $u \in B(x, r/16)$ *and* $v \in B(y, r/16)$

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Let $D = B(x,r) \cup B(y,r)$. Let $D_1 = B(x,r)$ and $D_3 = B(y,r)$ be disjoint. Then $\inf_{u \in D_1,\, z \in D_3} \nu(z-u) \geq \nu(2(x-y))$

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$$p_D(t,u,v) \ge tP^u(\tau_{D_1} > t)\, P^v(\tau_{D_3} > t) \inf_{u\in D_1, z\in D_3} \nu(z-u)$$

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Lemma

Let $D = B(0,1)$. *Let* $r \le 1$. *Let* $x \in D : \delta_D(x) < r/6$. *Denote* $x_0 = x/|x|$, $x_1 = x_0(1 - r/3)$ *and* $B_x = B(x_1, r/8)$. *There are* C, c *such that for* $t \le cV^2(r)$,

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Suppose that $t < t_0$ and choose r such that $t = cV^2(r)$ then for $x \in D$ with $\delta_D(x) < r/6$.

$$\int_{B_x} p_D(t,x,u)du \ge CP^x(\tau_D > t).$$

Assume that the same holds for $y \in D$

$$\int_{B_\cdot} p_D(t,x,u)du \geq CP^\cdot(\tau_D > t).$$

$$\begin{aligned}
p_D(3t,x,y) &\geq \int_{B_y}\int_{B_x} p_D(t,x,u)p_D(t,u,v)p_D(t,v,y)dudv \\
&\geq \inf_{u\in B_x, v\in B_y} p_D(t,u,v)\int_{B_x} p_D(t,x,u)du \int_{B_y} p_D(t,v,y)dv \\
&\geq Cp(t,c(x-y))P^x(\tau_D > t)P^y(\tau_D > t)\,.
\end{aligned}$$

Estimate of $p_D(t,x,y)$ for $D = B(0,1)^c$

Let $x, y \in D$. Let $t \le 1$. We choose r: $V(r) = \sqrt{t}$. First, assume $V^2(|x-y|/3) \ge t$, so $|x-y| \ge 3r$. We define

$$D_1 = B(x_0, r) \cap D, \quad x_0 = x/|x|$$

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We assume $0 < \delta_D(x) \leq r/3$. Note that $z \in D_2 = D \setminus (D_1 \cup D_2)$,

$$|z-y| \geq |x-y|/2,$$

$$\sup_{s<t,\, z\in D_2} p(s, z-y) \leq \sup_{s<t} p(t, (x-y)/2) =: q(t, (x-y)/2)$$

Moreover for $u \in D_1$, $z \in D_3$ we have
$|u - z| \geq |x - y|/2 - |x - u| - |x_0 - x| \geq |x - y|/18$, hence

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By Lemma

$$\begin{aligned} p_D(t, x, y) &\leq P^x(X_{\tau_{D_1}} \in D_2) \sup_{s<t,\, z \in D_2} p(s, z - y) \\ &+ (t \wedge E^x \tau_{D_1}) \sup_{u \in D_1,\, z \in D_3} \nu(z - u)\,, \end{aligned}$$

hence

$$p_D(t, x, y) \leq P^x(X_{\tau_{D_1}} \in D_2) q(t, (x - y)/2) + E^x \tau_{D_1} \nu((x - y)/18)\,,$$

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Next,

$$P^x(X_{\tau_{D_1}} \in D_2) \leq C \frac{E^x \tau_{D_1}}{V^2(r)}$$

and

$$E^x \tau_{D_1} \leq C V(r) V(\delta_D(x)).$$

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Recall that $V^2(r) = t$, so we obtain

$$\begin{aligned} p_D(t,x,y) &\leq C\frac{V(\delta_D(x))}{\sqrt{t}}(q(t,(x-y)/2) + t\nu((x-y)/18) \\ &\leq C\frac{V(\delta_D(x))}{\sqrt{t}}(p(t,c(x-y)), \end{aligned}$$

where we use the assumption on $q(t,x)$ and the Lévy density (if $V^2(|x-y|/3) \geq t$).

Next, we deal $V^2(|x-y|/3) \le t$. Then it is trivial that

$$p_D(2t, x, y) \le p(t, 0) P^x(\tau_D > t) \le p(t, 0) \frac{V(\delta_D(x))}{\sqrt{t}},$$

where the last step follows from one of the previous Lemmas. If $V^2(|x-y|/3) \le t$ then $p(t, 0) \le Cp(t, c(x-y))$.

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If $V^2(|x-y|/3) \le t$ then $p(t,0) \le Cp(t,c(x-y))$.
Combining all facts we obtain

$$p_D(t,x,y) \le C\left(\frac{V(\delta_D(x))}{\sqrt{t}} \wedge 1\right) p(t,c(x-y)),$$

where $x, y \in D$ $|x|, |y| \le R, \quad R > 1$ and the constant $C = C(R)$ increases with R.

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$$p_D(t,x,y) \le CP^x(\tau_D > t)p(t,c(x-y)),$$

where C might depend on R in the increasing way.