Brownian couplings and applications

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6th International Conference on Stochastic Analysis and Its Applications

Bedlewo, Poland, September 10 – 14, 2012

Plan of the talk

The method of coupling of reflecting Brownian motions (RBM) is a useful technique for proving results on various functionals associated to RBM.

In this talk, we will present two such couplings: the scaling coupling and the mirror coupling.

As an application of the scaling coupling, we will prove a monotonicity of the lifetime of reflecting Brownian motion with killing, which implies the validity of the Hot Spots conjecture of J. Rauch for a certain class of domains.

As applications of the mirror coupling, we will present the proof of the Laugesen-Morpurgo conjecture, and a unifying proof of the results of I. Chavel and W. Kendall on Chavel's conjecture.

Time-permitting, I will discuss some recent results on translation coupling and its applications.

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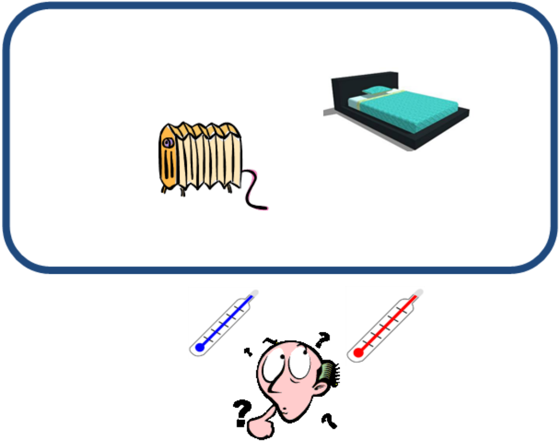

– *Where should I put the bed, to keep warm* ***in the long run****?*

Heuristics

Consider $u(t,x)$ the solution of the Neumann heat equation in a smooth bounded domain $D \subset \mathbb{R}^d$ with generic initial condition u_0.

Let x_t^+ be the hot spot at time t and x_t^- be the cold spot, i.e.

$$u\left(t,x_t^+\right) = \max_{x\in\overline{D}} u(t,x) \qquad \text{and} \qquad u\left(t,x_t^-\right) = \min_{x\in\overline{D}} u(t,x)$$

If the second Neumann eigenvalue λ_2 is simple, and φ_2 is a corresponding second Neumann eigenfunction, for large t we have

$$u(t,x) = \int_D u_0 + e^{-\lambda_2 t}\varphi_2(x)\int_D u_0\varphi_2 + R_2(t,x) \approx c_0 + c_1 e^{-\lambda_2 t}\varphi_2(x),$$

so x_t^+ and x_t^- are close to the maximum/minimum points of φ_2.

Hot spots (x_t^+) and cold spots (x_t^-) repel each other, so the distance between them tends to increase wrt t. In convex domains, the maximum distance is attained for points on the boundary.

Together with the above, this suggests the following.

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Conjecture 1 (Hot Spots conjecture)

Maxima and minima of second Neumann eigenfunctions of convex bounded domains are attained (only) on the boundary of the domain.

- B. Kawohl: true for balls, annuli, parallelipipeds in $\mathbb{R}^d$
- K. Burdzy and W. Werner: counterexample (non-convex domain): minimum inside the domain, maximum on the boundary
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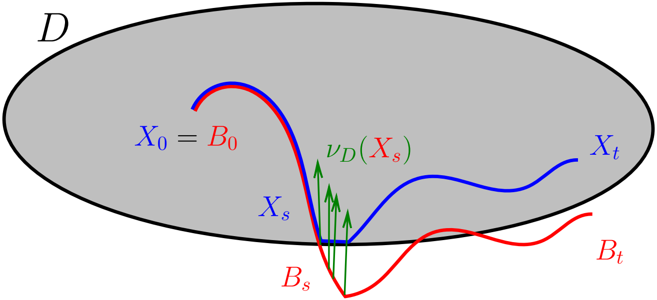
Definition 2 (Reflecting Brownian motion)

Reflecting Brownian motion in $D \subset \mathbb{R}^d$ starting at $x_0 \in \overline{D}$: a solution to

$$X_t = x_0 + B_t + \int_0^t \nu_D(X_s)\, dL_s^X, \qquad t \geq 0, \tag{1}$$

where B_t is a d-dimensional Brownian motion starting at origin, ν_D is the inward unit vector field on ∂D, L_t^X is the local time of X on ∂D.

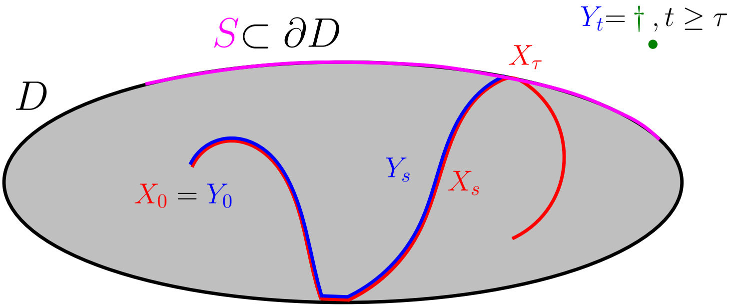


Definition 3 (Reflecting Brownian motion with killing)

Reflecting Brownian motion in D killed on hitting $S \subset \partial D$, starting at $x_0 \in \overline{D}$:

$$Y_t = \begin{cases} X_t, & t < \tau \\ \dagger, & t \geq \tau \end{cases}, \qquad (2)$$

where X_t is RBM in D starting at x_0, $\tau = \tau_S = \inf\{t > 0 : X_t \in S\}$ is the killing time, and $\dagger \notin D$ is the cemetery state.



Couplings of RBM

BM is invariant under translation, rotation/symmetry and scaling (almost).

This gives rise to:

- Synchronous coupling: $(B_t, B_t + v)$
- Mirror coupling: (B_t, RB_t)
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The above can be extended to the case of reflecting Brownian motion.

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Lemma 4 (“Multiplicative Skorokhod lemma” in the unit disk, MNP)

If B_t *is a 2-dimensional BM,* $M_t = 1 \vee \sup_{s \le t} |B_s|$ *and* $\alpha_t^{-1} = A_t = \int_0^t \frac{1}{M_s^2} ds$,

$$X_t = \frac{1}{M_{\alpha_t}} B_{\alpha_t}, \qquad t \ge 0$$

is a RBM in $U = \{z \in \mathbb{R}^2 : |z| < 1\}$.

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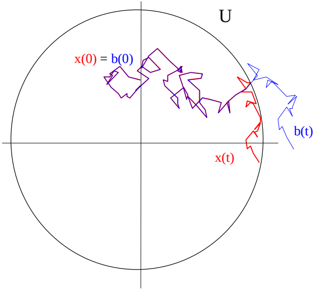

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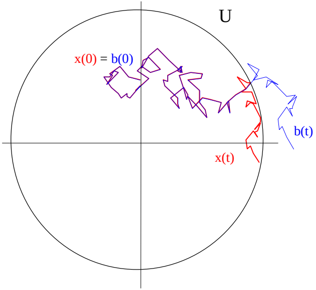
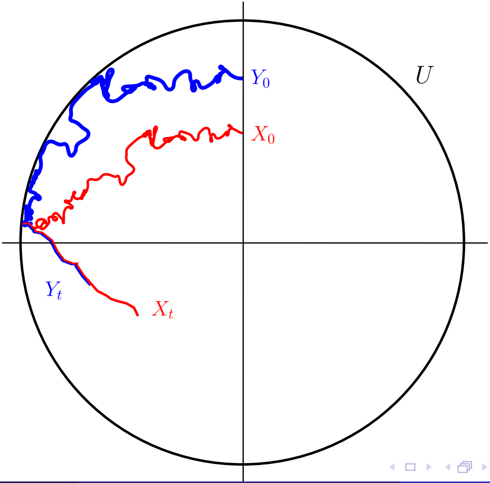


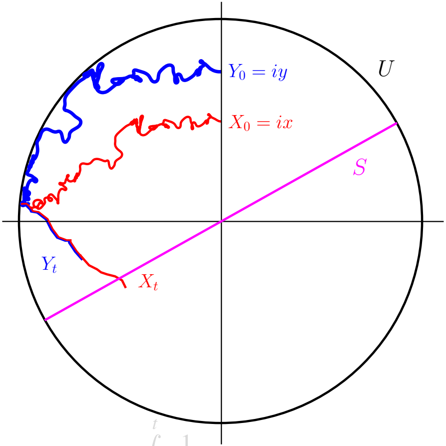
Proof: Itô formula with $f(x,y) = \frac{x}{y}$, B_t and M_t (and a time change).

Scaling coupling of RBM in U starting at $(xe^{i\theta}, ye^{i\theta})$ $(0 < x \le y \le 1)$:
a pair (X_t, Y_t), where X_t RBM in U starting at $xe^{i\theta}$, $Y_t = \dfrac{1}{M_{\alpha_t}} X_{\alpha_t}$,

$M_t = \frac{x}{y} \vee \sup_{s \le t} |X_s|$, and $\alpha_t^{-1} = A_t = \displaystyle\int_0^t \frac{1}{M_s^2} ds$.



Lifetime of RBM in the unit disk, killed on a diameter



$$M_t = \frac{x}{y} \vee \sup_{s \le t} |X_s| \le 1 \Longrightarrow A_t = \int_0^t \frac{1}{M_s^2} ds \ge t \Longrightarrow \alpha_t = A_t^{-1} \le t \Longrightarrow \tau^X = \alpha_{\tau^Y} \le \tau^Y$$

(τ^X, τ^Y denote the lifetime of X_t, Y_t killed on the diameter S).

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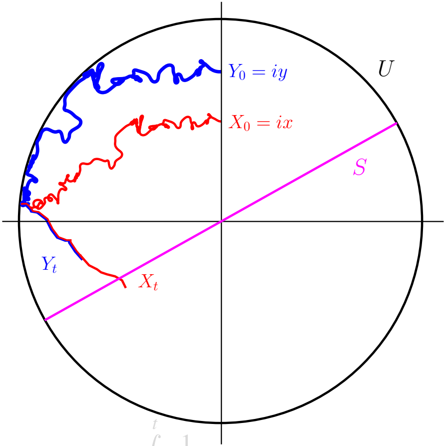

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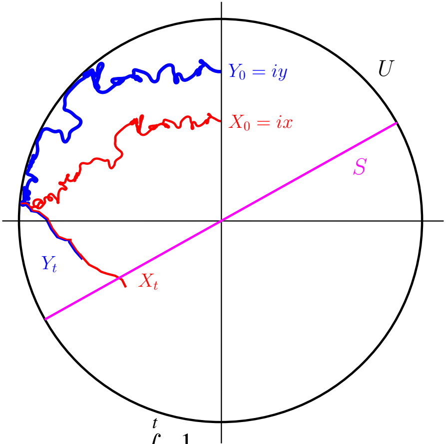

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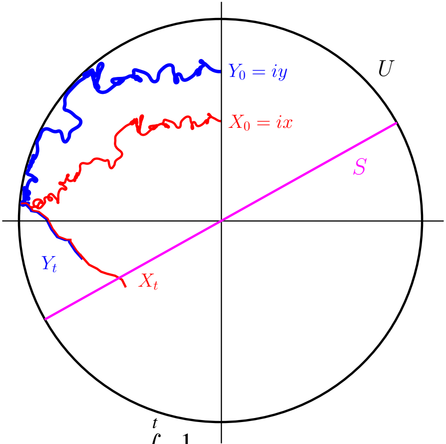

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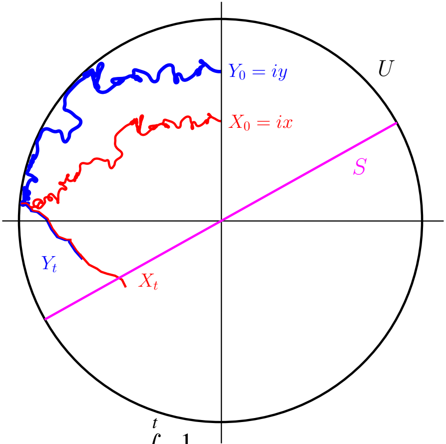

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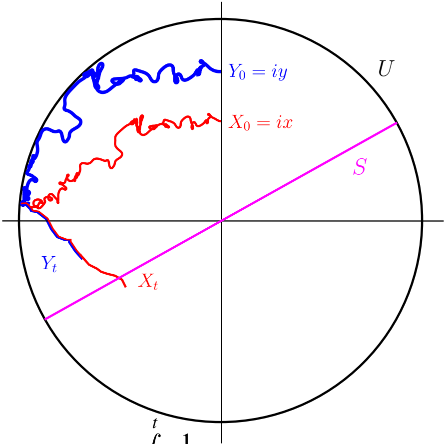

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Scaling coupling and applications

Corollary 5 (Monotonicity of lifetime in the disk)

For any $t > 0$, $P(\tau^x > t)$ is a radially increasing function in U (τ^x is the lifetime of RBM in U starting at x, killed on a diameter).

Remark: for t large, $P(\tau^x > t) \approx ce^{-\mu_1 t}\psi_1(x) = ce^{-\lambda_2 t}\varphi_2(x)$.

Theorem 6 (Monotonicity of antisymmetric second Neumann eigenfunctions)

If φ is a second Neumann eigenfunction of the Laplacian on U, antisymmetric with respect to a diameter, then φ is a radially monotone function.

Remark: any second Neumann eigenfunction is antisymmetric in the disk!

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The Hot Spots conjecture holds for the unit disk U, that is for any second Neumann eigenfunction φ of the laplacian on U we have

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The previous result is known (B. Kawohl, [6])... ☹

Conformal invariance of RBM + geometric characterization of a convex maps $\Rightarrow$ the same is true for any smooth bounded convex domain $D \subset \mathbb{R}^2$! ☺

Theorem 8 (MNP)

If $D \subset \mathbb{R}^2$ is a convex $C^{1,\alpha}$ domain $(0 < \alpha < 1)$, and at least one of the following hypothesis hold,

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ii) *D is symmetric with respect to the horizontal axis and the diameter to width ratio d_D/l_D is larger than $\frac{4j_0}{\pi} \approx 3.06$;*

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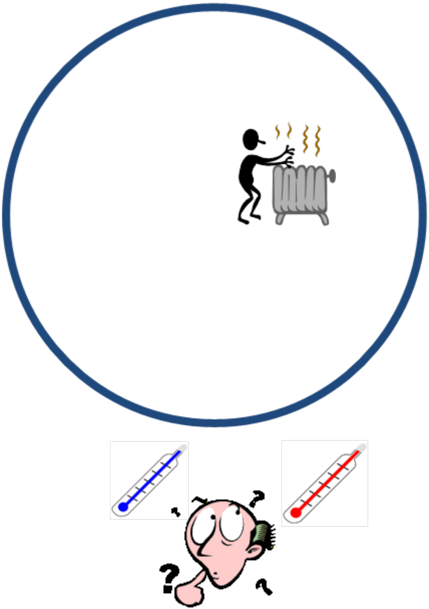

– *Where should I put the radiator, to feel warmest* ***at all times****?*

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Mirror coupling of BM/RBM

Was introduced by Kendall ([7]) and Cranston ([Cr]) (BM on manifolds), and developed by Burdzy et al. ([1], [2], [3], ...) (RBM in a smooth domains).

Loosely speaking, in both cases the idea is that the increments of one BM/RBM is the mirror image of the other.

A bit more precise: let X_t, Y_t be RBM is a smooth domain $D \subset R^d$, with driving BM B_t, Z_t, and consider the SDE:

$$Z_t = B_t - 2\int_0^t \frac{X_s - Y_s}{\|X_s - Y_s\|^2}(X_s - Y_s) \cdot dB_s. \quad (4)$$

Burdzy et al. proved the existence of a strong solution and pathwise uniqueness of the above SDE for $t < \tau = \inf\{s > 0 : X_s = Y_s\}$.

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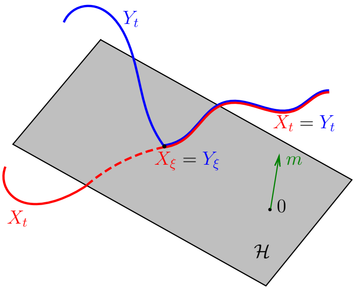
What does this mean?

For a unitary vector m, let $H(m) = I - 2mm'$ (reflection in the hyperplane through the origin and perpendicular to m), and define $G(x) = H\left(\frac{x}{||x||}\right)$ if $x \neq 0$ and $G(0) = I$.

$$(4) \Longleftrightarrow dZ_t = G\left(\frac{X_t - Y_t}{||X_t - Y_t||}\right) dW_t,$$

so, (4) says that the increments dZ_t and dB_t are mirror images wrt hyperplane of symmetry $\mathcal{M}_t$ between X_t and Y_t).



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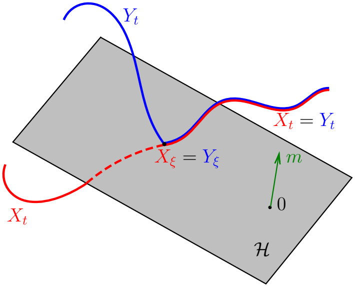

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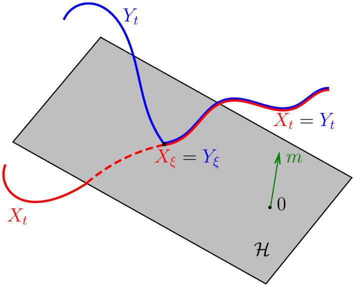

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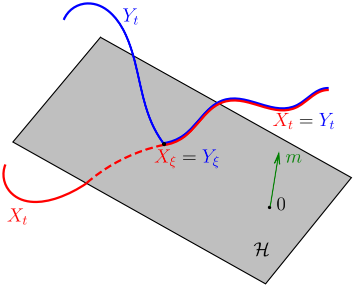

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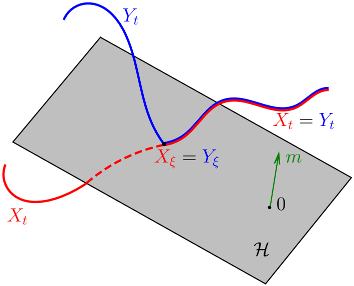

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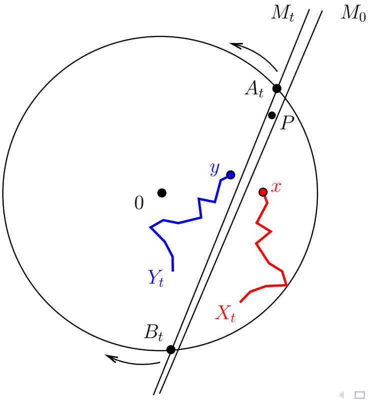
Lemma 10 ("Mirror $\mathcal{M}_t$ moves towards origin", MNP)

Let X_t, Y_t be a mirror coupling of RBM in $\mathbb{U}$ starting at $x, y \in \overline{\mathbb{U}}$, and let

$$\tau = \inf\{t > 0 : X_t = Y_t\} \quad \textit{and} \quad \tau_1 = \inf\{t > 0 : 0 \in \mathcal{M}_t\}.$$

For all times $t < \tau \wedge \tau_1$, the mirror $\mathcal{M}_t$ moves towards the origin, in such a way that if a point $P \in \mathbb{U}$ and the origin are separated by $\mathcal{M}_{t_1}$ for $t_1 \in [0, \tau \wedge \tau_1)$, then the point P and the origin are separated by $\mathcal{M}_{t_2}$ for all $t_2 \in [t_1, \tau \wedge \tau_1)$.



Theorem 11 (MNP)

For any points $x, y, z \in \overline{\mathbb{U}}$ such that $\|y\| \leq \|x\|$ and $\|x - z\| \leq \|y - z\|$, and any $t > 0$ we have:

$$p_{\mathbb{U}}(t, y, z) \leq p_{\mathbb{U}}(t, x, z). \tag{5}$$

Corollary 12

For any $x \in \mathbb{U} - \{0\}$, $r \in (0, \min\{\|x\|, 1 - \|x\|\})$ and $t > 0$ we have:

$$\int_{\partial \mathbb{U}} p_{\mathbb{U}}(t, x + ru, x)\, d\sigma(u) \leq p_{\mathbb{U}}(t, x + r\tfrac{x}{\|x\|}, x) \leq p_{\mathbb{U}}(t, x + r\tfrac{x}{\|x\|}, x + r\tfrac{x}{\|x\|}), \tag{6}$$

where σ is the normalized surface measure on $\partial\mathbb{U}$.

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Resolution of the Laugesen-Morpurgo conjecture

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Proof. $$\frac{d}{d\|x\|} p_{\mathbb{U}}(t,x,x) = \lim_{r \searrow 0} \frac{p_{\mathbb{U}}(t, x + r\frac{x}{\|x\|}, x + r\frac{x}{\|x\|}) - p_{\mathbb{U}}(t,x,x)}{r}$$

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$$\begin{aligned} \frac{d}{d\|x\|} p_{\mathbb{U}}(t,x,x) &= \lim_{r \searrow 0} \frac{p_{\mathbb{U}}(t, x + r\frac{x}{\|x\|}, x + r\frac{x}{\|x\|}) - p_{\mathbb{U}}(t,x,x)}{r} \\ &\geq \lim_{r \searrow 0} \frac{\int_{\partial \mathbb{U}} p_{\mathbb{U}}(t, x + ru, x)\, d\sigma(u) - p_{\mathbb{U}}(t,x,x)}{r} \end{aligned}$$

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$$p_{\mathbb{U}}(t,x,x) < p_{\mathbb{U}}(t,y,y), \tag{7}$$

for all $x,y\in\overline{\mathbb{U}}$ *with* $\|x\|<\|y\|$.

Proof.

$$\begin{aligned}
\frac{d}{d\|x\|}p_{\mathbb{U}}(t,x,x) &= \lim_{r\searrow 0}\frac{p_{\mathbb{U}}(t,x+r\frac{x}{\|x\|},x+r\frac{x}{\|x\|})-p_{\mathbb{U}}(t,x,x)}{r}\\
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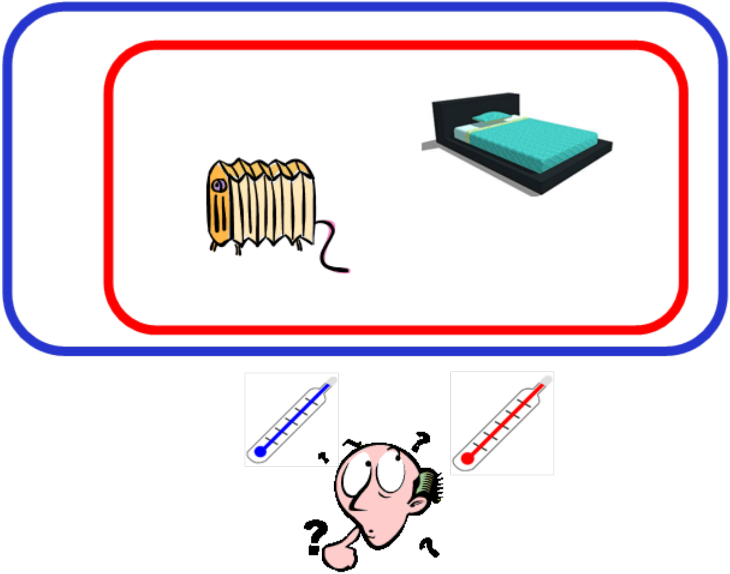
&= 0.
\end{aligned}$$



*– Are we going to be **warmer** or **colder** in a bigger apartment??*

Conjecture 14 (I. Chavel, 1986)

If $D_1 \subset D_2$ are convex domains then for all $t > 0$ and $x, y \in D_1$ we have

$$p_{D_1}(t,x,y) \geq p_{D_2}(t,x,y).$$

I. Chavel: TRUE, if D_2 is a ball centered at x (or y) and D_1 is convex (integration by parts).

W. S. Kendall: TRUE, if D_1 is a ball centered at x (or y) and D_2 is convex (coupling arguments).

Theorem 15 (Chavel + Kendall)

If $D_1 \subset D_2$ are convex domains then for all $t > 0$ and $x, y \in D_1$ we have

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Extension of the mirror coupling

Question: can one define a mirror coupling of Brownian motions living in *different domains* D_1 and D_2?

Answer: YES, for example if $D_1 = \mathbb{R}$ and $D_2 = (0, \infty)$:
Construction (Tanaka formula): X_t BM in $\mathbb{R}$, $Y_t = |X_t|$ RBM on $(0, \infty)$ and

$$dY_t = \operatorname{sgn}(X_t) dX_t + dL_t^X, \qquad t \geq 0. \tag{8}$$

YES, if:

- $D_1 = \mathbb{R}^2$ and D_2 half plane
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Extension of the mirror coupling

Question: can one define a mirror coupling of Brownian motions living in *different domains* D_1 and D_2?
Answer: YES, if $D_{1,2} \subset \mathbb{R}^d$ smooth, with non-tangential boundaries, and $D_1 \cap D_2$ convex

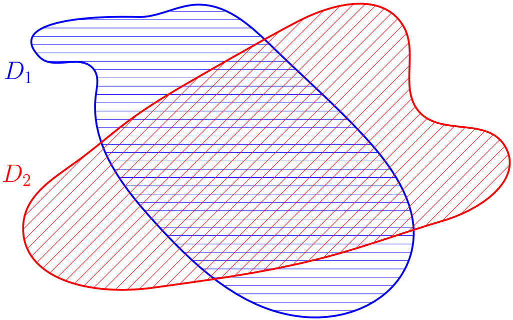


Figure: Typical domains for the extended mirror coupling.

An application of mirror coupling: a unifying proof of Chavel's conjecture

Theorem 16

If $D_1 \subset D_2 \subset \mathbb{R}^d$ are smooth and D_1 is a convex domain, then for all $t > 0$ and $x, y \in D_1$ we have

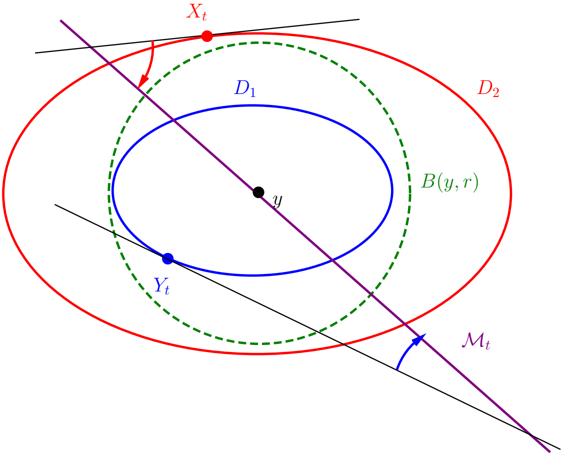
$$p_{D_1}(t,x,y) \geq p_{D_2}(t,x,y),$$

whenever there exists a ball B centered at either x or y such that $D_1 \subset B \subset D_2$.

Sketch of the proof

Consider a mirror coupling of RBM X_t, Y_t in D_2, D_1, starting at $X_0 = Y_0 = x$.

For all $t > 0$, the mirror $\mathcal{M}_t$ of the coupling cannot separate Y_t and y.



$$\|Y_t - y\| \leq \|X_t - y\|, \qquad t > 0 \qquad \Longrightarrow \qquad p_{D_1}(t, x, y) > p_{D_2}(t, x, y). \quad \square$$

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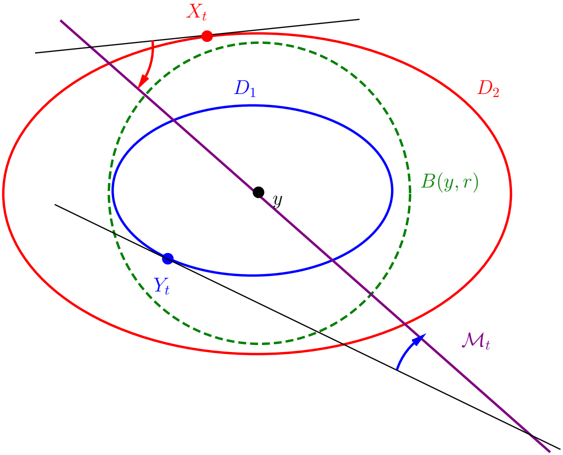

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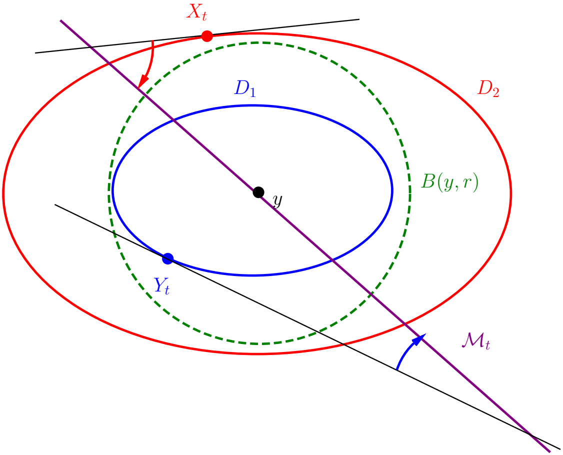

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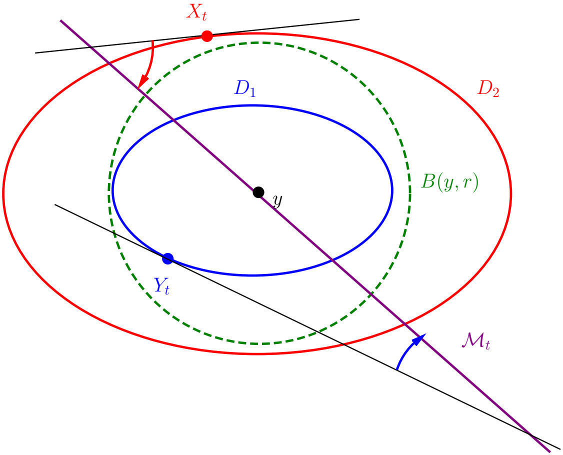

$\|Y_t - y\| \le \|X_t - y\|, \qquad t > 0 \qquad \Longrightarrow \qquad p_{D_1}(t,x,y) > p_{D_2}(t,x,y)$. □

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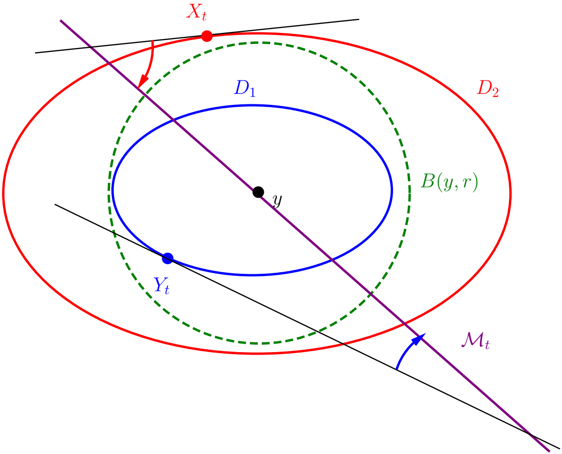

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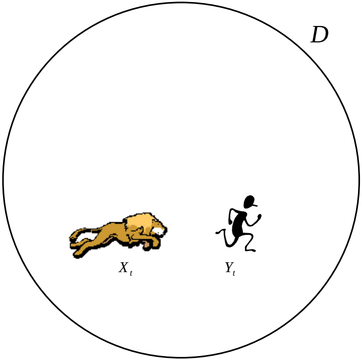

Given $x, y \in D$, does there exist X, Y with $X_0 = x$, $Y_0 = y$, $|X_t'| = |Y_t'| = 1$ and

$$dist(X_t, Y_t) > c, \text{ for all } t > 0?$$

(Littlewood, Besicovitch, Croft, Bollobas et. al., Nahin, ...)

Shy couplings (Benjamini, Burdzy and Chen – PTRF, 2007)

ε-shy coupling: $P(dist(X_t, Y_t) > \varepsilon \text{ for all } t \geq 0) > 0$.

RBM case: no co-adapted shy coupling in

- bounded convex planar domains with C^2 boundary, without line segments (BBC, 2007).
- bounded convex domains in $\mathbb{R}^d$, without line segments in the boundary if $d \geq 3$ (Kendall, 2009).
- bounded CAT(0) spaces with boundary satisfying uniform exterior sphere and interior cone conditions, e.g. simply connected bounded planar domains with C^2 boundary (Bramson, Burdzy & Kendall, preprint).

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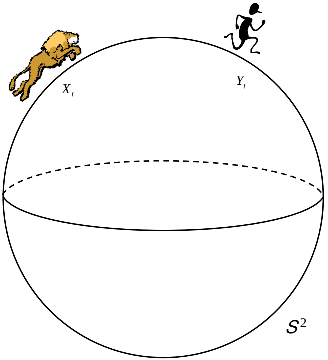
BMan and BLion problem on $\mathcal{S}^2$ (joint with I. Popescu)



Theorem 17

For any $X_0, Y_0 \in \mathcal{S}^2$, there exist couplings of BM on $\mathcal{S}^2$ s.t. for all $t \geq 0$ we have

a) $|X_t - Y_t| = \sqrt{4 - |y+x|^2 e^{-t}} \nearrow 2$ *(distance-increasing coupling)*

b) $|X_t - Y_t| = |y-x| e^{-t/2} \searrow 0$ *(distance-decreasing coupling)*

c) $|X_t - Y_t| = |y-x| = const$ *(fixed-distance coupling= "translation coupling").*

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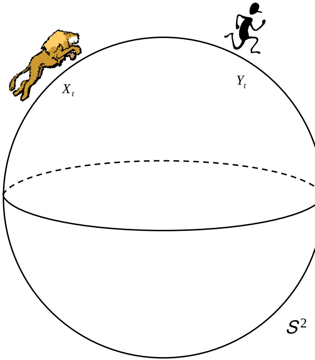

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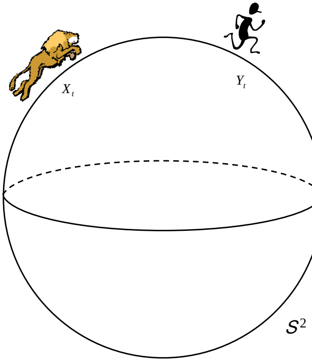

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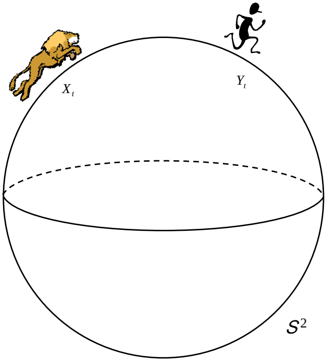

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