Ricci flow, Brownian motion and entropy

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Joint work with K. Kuwada, H. Guo and A. Thalmaier

- $M = d$-dim. smooth manifold

- $(g(t))_{t\in[0,T]}$ smooth family of Riemannian metrics on M,

  e.g. $\frac{\partial g}{\partial t} = \pm 2\,\mathrm{Ric}$, (backward) Ricci flow

- $(M, g(t))$ complete for all $t \in [0, T]$

Definition (Arnaudon, Coulibaly, Thalmaier 2008)

A process $(X_t)_{t\in I}$ *is a* **Brownian motion** *if* $\forall f \in \mathcal{C}_b^{1,2}([0,T]\times M)$

$$df(t,X_t) = \left[\frac{\partial f}{\partial t} + \Delta_{g(t)} f\right](t,X_t)dt + \underbrace{dM_t}_{\textit{martingale}}$$

(The choice of Δ instead of $\frac{1}{2}\Delta$ is better adapted to Ricci flow.)

Let $u(t,\cdot)$ be the density of X_t, then

$$\frac{\partial u}{\partial t} = \Delta_{g(t)} u \qquad \underbrace{-\frac{1}{2}\operatorname{tr}\left(\frac{\partial g}{\partial t}\right) u}_{\text{change of volume element}}$$

1. Criterion for non-explosion (Kuwada, Philipowski '11):
   If $\exists C \in \mathbb{R}$:
   $$\frac{\partial g}{\partial t} \leq 2\,\mathrm{Ric}_{g(t)} + Cg(t),$$
   then Brownian motion does not explode.

2. Application to a new entropy formula
   (Guo, Philipowski, Thalmaier '12)

Idea to prove non-explosion: Fix a point $o \in M$ and let

$$\rho(t,x) := d_{g(t)}(o,x).$$

Since $(M, g(t))$ is complete for each $t \in [0, T]$,

$$X \text{ explodes at some time } \zeta \leq T \Leftrightarrow \rho(t, X_t) \text{ is unbounded on } [0,\zeta).$$

Therefore study the one-dimensional process $\rho(t, X_t)$.

For smooth functions $f : [0, T] \times M \to \mathbb{R}$ we have Itô's formula

$$df(t, X_t) = \left[\frac{\partial f}{\partial t} + \Delta_{g(t)} f\right] (t, X_t) dt + dM_t,$$

where M is a local martingale with

$$d\langle M \rangle_t = |\nabla f(t, X_t)|^2 dt.$$

The function ρ is smooth everywhere except on

$$\{(t, x) \in [0, T] \times M \mid x = o \text{ or } x \in \mathrm{Cut}_{g(t)}(o)\},$$

and $|\nabla \rho| = 1$.

For all $t \in [0, T]$, $\mathrm{Cut}_{g(t)}$ has volume 0, hence

$$\mathbb{P}\left[X_t \in \mathrm{Cut}_{g(t)}\right] = 0.$$

Guess:

$$d\rho(t, X_t) = \left[\frac{\partial \rho}{\partial t} + \Delta_{g(t)}\rho\right](t, X_t)dt + \underbrace{d\beta_t}_{\text{1-dim Brownian motion}}$$

This is not true!

Counterexample: $M = S^1$, $g(t) =$ standard metric $\forall t$
$\Rightarrow \Delta\rho = 0$ a.e., so we would expect

$$d\rho(t, X_t) = d\beta_t$$

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$$\underbrace{d\rho(t, X_t)}_{\in [0, 2\pi]} = \underbrace{d\beta_t}_{\text{unbounded}}$$

Reason: Reflection of the process $\rho(t, X_t)$ at o and $\text{Cut}_{g(t)}(o)$.
But the formula is true as long as $X_t \notin \{o\} \cup \text{Cut}_{g(t)}(o)$.

In dimension $d \geq 2$, the point o is never hit by Brownian motion, but $\mathrm{Cut}_{g(t)}(o)$ is hit in general, hence we need a correction term.

Theorem (Kendall '87 fixed metric; Kuwada, Philipowski '11 general case)

$\exists$ non-decreasing process L which increases only when $X_t \in \mathrm{Cut}_{g(t)}(o)$, such that

$$d\rho(t, X_t) = \left[\frac{\partial \rho}{\partial t} + \Delta_{g(t)}\rho\right](t, X_t)dt + d\beta_t - dL_t$$

Thanks to this Itô formula, it suffices to control the drift term $\frac{\partial \rho}{\partial t} + \Delta_{g(t)}\rho$.

Theorem (Kuwada, Philipowski '11)

If

$$\frac{\partial g}{\partial t} \leq 2\,\mathrm{Ric}_{g(t)} + Cg(t),$$

then $\exists K < \infty : \forall (t, x) \in [0, T] \times M$
such that $x \notin \mathrm{Cut}_{g(t)}(o)$ *and* $\rho(t, x) \geq 1$,

$$\left[\frac{\partial \rho}{\partial t} + \Delta_{g(t)}\rho\right](t, x) \leq K + C\rho(t, x).$$

Consequently, Brownian motion cannot explode.

Ricci flow and entropy formulae

Let M be compact,

$$\frac{\partial g}{\partial t} = -2\,\mathrm{Ric}$$

and u a non-negative solution of

$$\frac{\partial u}{\partial t} = -\Delta u + Ru.$$

Let

$$\mathrm{Ent}(t) := \int_M (u \log u)(t, y)\,\mathrm{vol}_{g(t)}(dy)$$

be the Boltzmann-Shannon entropy of $u(t, \cdot)$ with respect to the measure $\mathrm{vol}_{g(t)}$.

$$\begin{aligned}
\mathrm{Ent}(t) &= \int_M (u \log u)(t,y)\,\mathrm{vol}_{g(t)}(dy) \\
\mathrm{Ent}'(t) &= \int_M \Big(\Big(|\nabla(\log u)|^2 + R\Big)u\Big)(t,y)\,\mathrm{vol}_{g(t)}(dy) \\
&= \text{Perelman's } \mathcal{F}\text{-functional}, \\
\mathrm{Ent}''(t) &= 2\int_M \Big(|\mathrm{Ric} + \mathrm{Hess}(\log u)|^2\, u\Big)(t,y)\,\mathrm{vol}_{g(t)}(dy) \\
&\geq 0.
\end{aligned}$$

Proof: Integration by parts (M is compact).

Moreover,

$$\mathrm{Ent}''(t) = 0 \quad \Leftrightarrow \quad \mathrm{Ric} = -\mathrm{Hess}(\log u),$$

i.e. g is a gradient steady soliton (constant up to diffeomorphism)

Consequence (Perelman): Any periodic (up to diffeomorphism) solution (steady breather) is a gradient steady soliton.

Problem: If M is not compact, all this does not work (integrals may not exist; even if they exist, integration by parts need not be feasible).

Idea: (Guo, Philipowski, Thalmaier (2012)): Instead of

$$\begin{aligned}\frac{\partial g}{\partial t} &= -2\,\mathrm{Ric}\\ \frac{\partial u}{\partial t} &= -\Delta u + Ru\\ \mathrm{Ent}(t) &= \int_M (u\log u)(t,y)\,\mathrm{vol}_{g(t)}(dy)\end{aligned}$$

consider

$$\begin{aligned}\frac{\partial g}{\partial t} &\leq 2\,\mathrm{Ric}\\ \frac{\partial u}{\partial t} &= -\Delta u\\ \mathcal{E}(t) &:= E\left[(u\log u)(t,X_t)\right],\end{aligned}$$

where $(X_t)_{t\geq 0}$ is a $(g(t))_{t\geq 0}$-Brownian motion.

$$\begin{aligned} \frac{\partial g}{\partial t} &\leq 2\,\mathrm{Ric} \\ \frac{\partial u}{\partial t} &= -\Delta u \\ \mathcal{E}(t) &:= E\left[(u \log u)(t, X_t)\right] \end{aligned}$$

Advantages:

- $\mathcal{E}(t)$ is always well-defined (possibly $+\infty$), and finite in most cases
- Instead of integration py parts, use Itô's formula to compute $\mathcal{E}'(t)$ and $\mathcal{E}''(t)$

To compute $\mathcal{E}'(t)$ and $\mathcal{E}''(t)$ using Itô's formula, we need:

$$\begin{aligned}
\left(\frac{\partial}{\partial t} + \Delta_{g(t)}\right)(u \log u) &= \frac{|\nabla u|^2}{u} \\
\left(\frac{\partial}{\partial t} + \Delta_{g(t)}\right)\left(\frac{|\nabla u|^2}{u}\right) &= 2u|\operatorname{Hess}\log u|^2 \\
&\quad + u\left(2\operatorname{Ric} - \frac{\partial g}{\partial t}\right)(\nabla \log u, \nabla \log u) \\
&\geq 0 \qquad \text{if } \frac{\partial g}{\partial t} \leq 2\operatorname{Ric}
\end{aligned}$$

Theorem (Guo, Philipowski, Thalmaier '12)

Under mild assumptions (which guarantee that certain local martingales are true martingales),

$$\begin{aligned}
\mathcal{E}'(t) &= E\left[\frac{|\nabla u|^2}{u}(t, X_t)\right] \geq 0, \\
\mathcal{E}''(t) &= E\left[(A+B)(t, X_t)\right], \ \textit{where} \\
A &= 2u|\mathrm{Hess}\log u|^2 \geq 0, \\
B &= u\left(2\,\mathrm{Ric} - \frac{\partial g}{\partial t}\right)(\nabla \log u, \nabla \log u) \\
&\geq 0 \ \textit{in the case of backward super Ricci flow}
\end{aligned}$$

Hence, under backward super Ricci flow $\mathcal{E}$ is non-decreasing and convex.

Proof: By Itô's formula,

$$\begin{aligned} d(u \log u)(t, X_t) &\overset{\text{m}}{=} \left(\frac{\partial}{\partial t} + \Delta_{g(t)}\right)(u \log u)(t, X_t)dt \\ &= \frac{|\nabla u|^2}{u}(t, X_t)dt, \end{aligned}$$

so that

$$\begin{aligned} \mathcal{E}'(t) &= \frac{d}{dt} E[(u \log u)(t, X_t)] \\ &= E\left[\frac{|\nabla u|^2}{u}(t, X_t)\right]. \end{aligned}$$

Same argument for $\mathcal{E}''(t)$.

Application to ancient solutions of the heat equation

Suppose a solution u of the backward heat equation

$$\frac{\partial u}{\partial t} = -\Delta_{g(t)} u$$

is defined for all $t \geq 0$. Then it can be regarded as an ancient solution of the forward heat equation, and the monotonicity and convexity of $\mathcal{E}$ imply:

1. If $\mathcal{E}(t)$ grows sublineary, i.e. if $\lim_{t\to\infty} \frac{\mathcal{E}(t)}{t} = 0$, u must be constant.
2. If $\mathcal{E}(t)$ is exactly linear, i.e. if $\mathcal{E}''(t) \equiv 0$, then u has the form $u(t, x) = \psi(x)\varphi(t)$.

Proof (1): Since $\mathcal{E}$ is convex, the condition $\lim_{t\to\infty} \frac{\mathcal{E}(t)}{t} = 0$ implies that $\mathcal{E}$ is constant. Therefore

$$\mathcal{E}'(t) = E\left[\frac{|\nabla u|^2}{u}(t, X_t)\right] \equiv 0,$$

so that u is constant.

Proof (2): If $\mathcal{E}''(t) \equiv 0$, we have

$$\mathrm{Hess}(\log u) \equiv 0,$$

so that

$$\begin{aligned} \frac{\partial \log u}{\partial t} &= \frac{1}{u}\Delta u \\ &= \Delta(\log u) + \frac{1}{u^2}|\nabla u|^2 \\ &= |\nabla \log u|^2. \end{aligned}$$

This implies

$$\log u(t,y) - \log u(0,y) = -\int_0^t \underbrace{|\nabla \log u|^2(s)}_{\text{does not depend on } y} ds,$$

so that

$$u(t,y) = u(0,y)\exp\left(-\int_0^t |\nabla \log u|^2(s) ds\right).$$