Fractional Laplacian with singular drift

Tomasz Jakubowski

Będlewo, 11.06.2012

Talk is based on two papers:

- K. Bogdan, T. J., *Estimates of heat kernel of fractional Laplacian perturbed by gradient operators.*, Comm. Math. Phys., 271(1):179–198, 2007.

- T. J., *Fractional Laplacian with singular drift.* Studia Math. 207 (3) (2011)

Fractional Laplacian

- Let $d \in \mathbb{N}$ and $\alpha \in (0,2)$. We define $p\colon (0,\infty) \times \mathbb{R}^d \to \mathbb{R}$

$$\int_{\mathbb{R}^d} p(t,z) e^{iz\cdot\xi} dz = e^{-t|\xi|^\alpha}, \quad \xi \in \mathbb{R}^d, \quad t > 0,$$

$p(t,x,y) = p(t,y-x)$ - density of isotropic α-stable process.

- Fractional Laplacian - generator of stable semigroup

$$\Delta^{\alpha/2} f(x) = \lim_{t\to 0^+} \frac{1}{t} \int_{\mathbb{R}^d} p(t,y-x)(f(y)-f(x))\, dy$$
$$= \lim_{\varepsilon\to 0^+} c \int\limits_{|y|>\varepsilon} \frac{f(x+y)-f(x)}{|y|^{d+\alpha}} dy\,, \quad f \in C_c^\infty(\mathbb{R}^d)\,.$$

Problem

Estimate the density $\tilde{p}$ *of the semigroup generated by* $\Delta^{\alpha/2} + b \cdot \nabla$

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We assume $1 < \alpha < 2$

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$b\in\mathcal{K}_d^{\alpha-1}$ *iff*

$$\lim_{t\to 0}\sup_{x\in\mathbb{R}^d}\int_0^t\int_{\mathbb{R}^d}s^{-1/\alpha}p(s,x,y)|b(y)|\,dy\,ds=0\,.$$

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Theorem (K. Bogdan, T.J., 2007)

There is continuous transition density $\tilde{p}(t,x,y)$ such that (weakly)

$$\lim_{t \to 0} \frac{\tilde{P}_t f(x) - f(x)}{t} = \Delta^{\alpha/2} f(x) + b(x) \cdot \nabla f(x) \,,$$

where $\tilde{P}_t f(x) = \int_{\mathbb{R}^d} \tilde{p}(t,x,y) f(y)\, dy$.

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where $\tilde{P}_t f(x) = \int_{\mathbb{R}^d} \tilde{p}(t,x,y) f(y)\,dy$. *Furthermore,*

$$C^{-1} p(t,x,y) \le \tilde{p}(t,x,y) \le Cp(t,x,y)\,, \quad 0 < t < t_0\,,\ x,y \in \mathbb{R}^d\,,$$

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$$C^{-1} p(t,x,y) \le \tilde{p}(t,x,y) \le C p(t,x,y)\,, \quad 0 < t < t_0\,,\ x,y \in \mathbb{R}^d\,,$$

where $C = C(d,\alpha,b,t_0)$, and $C \to 1$ as $t_0 \to 0$.

Consider:

- $\{\partial/\partial t - \Delta^{\alpha/2}\} - A_b$, where $A_b = b(x) \cdot \nabla_x$,

Perturbation series: Informal introduction

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$$Pf(s,x) = \int_0^t \int_{\mathbb{R}^d} p(t-s,x,z)f(s,z)dzds\,, \quad f \in C_0^\infty(R_+ \times \mathbb{R}^d)\,.$$

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We inductively define the integral kernel p_n of $(PA_b)^n P$,

functions p_n

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Hence, $p_0 := p$, and for $n \geq 1$,

$$p_n(t,x,y) := \int_0^t \int_{\mathbb{R}^d} p_{n-1}(t-s,x,z) b(z) \cdot \nabla_z p(s,z,y)\, dz\, ds\,.$$

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$\tilde{p}(t,x,y)$ shoud be the density of $\tilde{P}$

estimate of p_1

Lemma

Let $b \in \mathcal{K}_d^{\alpha-1}$. For every $t > 0$ there is $C = C(d, \alpha, b, t)$ such that

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$C \to 0$ as $t \to 0$.

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tools:

- $p(t,x) \approx t^{-d/\alpha} \wedge \frac{t}{|x|^{d+\alpha}}$
- transference property

$$|\nabla_x p(t,x)| \approx |x| \left(t^{-\frac{d+2}{\alpha}} \wedge \frac{t}{|x|^{(d+2)+\alpha}} \right) \le cp(t,x) \left(|x|^{-1} \wedge t^{-1/\alpha} \right)$$

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- 3P inequality

$$p(s, x)p(t, y) \le C\,p(s+t, x+y)(p(s, x) + p(t, y))$$

further developments

- Green function estimates:

  K. Bogdan, T.J., *Estimates of the Green Function for the Fractional Laplacian Perturbed by Gradient*, Potential Anal (2012) 36:455—481.

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- Other classes of drifts (depending also on time):

  T. J., K. Szczypkowski, *Time-dependent gradient perturbations of fractional Laplacian.* J. Evol. Equ., 10(2):319–339, 2010.

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We will consider drifts allowing singularities like

$$|b(z)| \approx |z|^{1-\alpha}$$

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- local in time outcomes
  - D. G. Aronson, 1968 - *Non-negative solutions of linear parabolic equations*.
  - Q. Zhang, 1996 - *Gaussian bounds for the fundamental solutions of* $\nabla(A\nabla u) + B\nabla u - u_t = 0$.
  - V. Liskevich and Y. Semenov, 2000 - *Estimates for fundamental solutions of second-order parabolic equations*.

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- global in time outcomes
  - V. Liskevich and Qi. S. Zhang, 2004 - *Extra regularity for parabolic equations with drift terms*.
  - Qi. S. Zhang, 2004 - A strong regularity result for parabolic equations.

classical case $-\Delta + b(x)\nabla_x$

V. Liskevich and Qi. S. Zhang, 2004

- There exists a constant B such that

$$\int_{\mathbb{R}^d} |b(x)|^2 \phi(x)^2 \le B \int_{\mathbb{R}^d} |\nabla \phi(x)|^2 dx\,, \quad \phi \in C_0^\infty(\mathbb{R}^d)$$

- There exists a constant $\delta > 0$ such that

$$\int_{\mathbb{R}^d} |\operatorname{div} b(x)| \phi(x)^2 \le (2-\delta) \int_{\mathbb{R}^d} |\nabla \phi(x)|^2 dx\,, \quad \phi \in C_0^\infty(\mathbb{R}^d)$$

- $$\sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{(\operatorname{div} b)_-(y)}{|x-y|^{d-2}} dy \quad \text{is sufficiently small}$$

Example in $\mathbb{R}^2$

$$b(x) = \left(\frac{x_2}{|x|^2}, \frac{-x_1}{|x|^2} \right), \qquad |b(x)| = |x|^{-1} = |x|^{1-2}$$

Theorem

There exist positive constants c_1 and c_2 such that, for any $x, y \in \mathbb{R}^d$ and $t > 0$

$$g(t,x,y) \leq \frac{c_1}{t^{d/2}} \exp\left(-\frac{c_2|x-y|^2}{t}\right),$$

where g is the density of the semigroup generated by $\Delta + b(x) \cdot \nabla$

α-stable case

(Conditions on b)

(1) *There is a constant C_b such that for all $s > 0$,*

$$\sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} s^{-1/\alpha} p(s,x,y)|b(y)|dy \le C_b s^{-1},$$

(2) $\operatorname{div} b = 0$ *in the sense of distributions*

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(recall that b belongs to Kato class $\mathcal{K}_d^{\alpha-1}$ if)

$$\lim_{t\to 0}\sup_{x\in\mathbb{R}^d}\int_0^t\int_{\mathbb{R}^d} s^{-1/\alpha} p(s,x,y)|b(y)|dyds = 0$$

Morrey space

Condition (1) is equivalent to

$$\sup_{t>0} \sup_{x\in\mathbb{R}^d} t^{-d+\alpha-1} \int_{B(x,t)} |b(y)|dy \leq C \qquad (\star)$$

- If b satisfies $(\star)$, it means that b belongs to the Morrey space $M_1^{1-\alpha}$.
- The best constant C in $(\star)$ is denoted by $||b||_{M_1^{1-\alpha}}$.

estimating p_1

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Condition (1) on b does not assure convergence of p_1:

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Condition (1) on b does not assure convergence of p_1:

$$|p_1(t,x,y)| \le c_1 \int_0^t \int_{\mathbb{R}^d} p(t-s,x,z)|b(z)|s^{-1/\alpha}p(s,z,y)dzds$$

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Condition (1) on b does not assure convergence of p_1:

$$\begin{aligned}|p_1(t,x,y)| &\leq c_1 \int_0^t \int_{\mathbb{R}^d} p(t-s,x,z)|b(z)|s^{-1/\alpha}p(s,z,y)dzds \\ &\leq cp(t,x,y)\int_0^t [(t-s)^{1/\alpha-1}s^{-1/\alpha} + s^{-1}]ds = \infty\end{aligned}$$

applying condition div $b = 0$

$$p_1(t,x,y) = \int_0^{t/2} \int_{\mathbb{R}^d} p(t-s,x,z) b(z) \cdot \nabla_z p(s,z,y) dz ds$$
$$\int_{t/2}^{t} \int_{\mathbb{R}^d} p(t-s,x,z) b(z) \cdot \nabla_z p(s,z,y) dz ds$$

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$$
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= \int_0^{t/2}\int_{\mathbb{R}^d} \nabla_z p(t-s,x,z)\cdot b(z)p(s,z,y)dzds \\
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applying condition div $b = 0$

$$
\begin{aligned}
p_1(t,x,y) &= \int_0^{t/2}\int_{\mathbb{R}^d} p(t-s,x,z)b(z)\cdot\nabla_z p(s,z,y)dzds \\
&\int_{t/2}^{t}\int_{\mathbb{R}^d} p(t-s,x,z)b(z)\cdot\nabla_z p(s,z,y)dzds \\
&= \int_0^{t/2}\int_{\mathbb{R}^d} \nabla_z p(t-s,x,z)\cdot b(z)p(s,z,y)dzds \\
&\int_{t/2}^{t}\int_{\mathbb{R}^d} p(t-s,x,z)b(z)\cdot\nabla_z p(s,z,y)dzds \\
&\le Cp(t,x,y),
\end{aligned}
$$

where C does not depend on t, x, y.

remark on div $b = 0$

- quasi geostrophic equation

$$\phi_t = \Delta^{\alpha/2}\phi - u \cdot \nabla\phi,$$
$$u = (u_1, u_2) = \left(-\frac{\partial\Psi}{\partial x_2}, \frac{\partial\Psi}{\partial x_1}\right),$$
$$\Delta^{1/2}\Psi = \phi.$$

[Constantin, Wu, 1999], [Schonbeck, Schonbeck, 2003], [Córdoba, Córdoba, 2004]

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[Constantin, Wu, 1999], [Schonbeck, Schonbeck, 2003], [Córdoba, Córdoba, 2004]

- pseudodifferential equation

$$u_t = \Delta^{\alpha/2}u - b \cdot \nabla u,$$
$$\operatorname{div} b = 0,$$
$$u(0, x) = u_0(x).$$

[Constantin, Wu, 2009], [Caffarelli, Vasseur, 2010], [Friedlander, Vico, 2011], [Silvestre, 2011]

estimating p_2

$$p_2(t,x,y) = \int_0^t \int_{\mathbb{R}^d} p_1(s,x,z) b(z) \cdot \nabla_z p(t-s,z,y) dz ds$$

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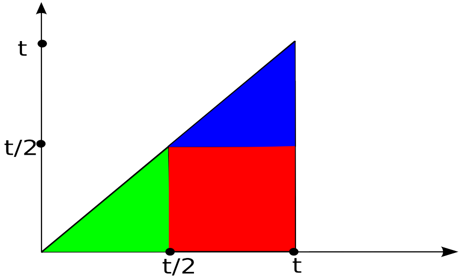


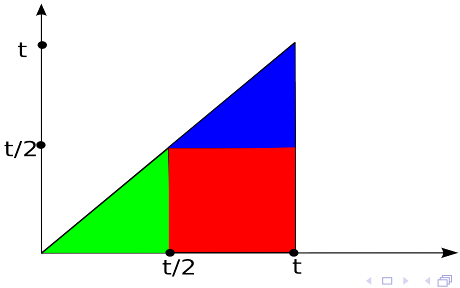
$$p_2(t,x,y)$$

$$= \int_0^{t/2}\int_{\mathbb{R}^d} p_1(s,x,z)b(z)\cdot\nabla_z p(t-s,z,y)dzds$$

$$+ \int_{t/2}^{t}\int_{\mathbb{R}^d} \nabla_z p(s,x,z)\cdot b(z)p_1(t-s,z,y)dzds$$

$$+ \int_{t/2}^{t}\int_0^{t/2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} p(r,x,w)\left[\nabla_z\cdot(b(w)\cdot\nabla_w p(s-r,w,z)b(z))\right]$$

$$p(t-s,z,y)dzdwdrds$$



partition of the simplex

$$S_n(a, b) := \{(s_1, s_2, \ldots, s_n) \in \mathbb{R}^n \colon a \leq s_1 \leq s_2 \leq \ldots \leq s_n \leq b\} \,.$$

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$$S_n(0,t) = S_n(0,t/2) \cup \left(\bigcup_{k=1}^{n-1} S_{n-k}(0,t/2) \times S_k(t/2,t) \right) \cup S_n(t/2,t)\,,$$

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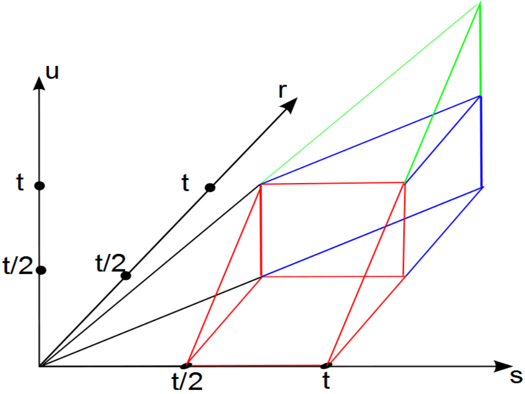

estimating p_n

Lemma

For $n \geq 2$ we have

$$p_n(t,x,y)$$
$$= \int_0^{t/2} \int_{\mathbb{R}^d} p_{n-1}(s,x,z) b(z) \cdot \nabla_z p(t-s,z,y) dzds$$
$$+ \int_{t/2}^{t} \int_{\mathbb{R}^d} \nabla_z p(s,x,z) \cdot b(z) p_{n-1}(t-s,z,y) dzds$$
$$+ \sum_{k=0}^{n-2} \int_{t/2}^{t} \int_0^{t/2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} dzdwdrds$$
$$p_{n-2-k}(r,x,w) \left[\nabla_z \cdot \left(b(w) \cdot \nabla_w p(s-r,w,z) b(z) \right) \right] p_k(t-s,z,y)$$

Motzkin numbers

Motzkin number M_n represents the number of different ways of drawing non-intersecting chords on a circle between n points

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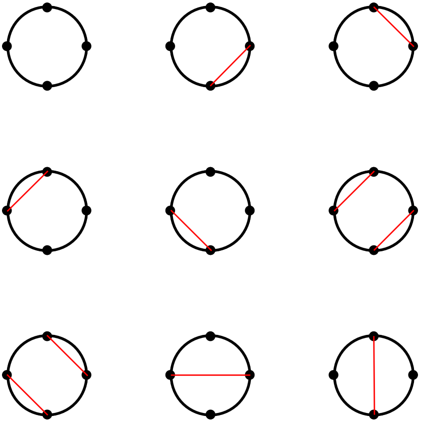

Main estimate

- generating function

$$M(x) = \sum_{n=0}^{\infty} M_n x^n = \frac{1 - x - \sqrt{1 - 2x - 3x^2}}{2x^2},$$

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Lemma

There is a constant C such that for all $t > 0$, $x, y \in \mathbb{R}^d$ and $n \geq 1$,

$$|p_n(t,x,y)| \leq M_n C^n p(t,x,y).$$

$$M_n = M_{n-1} + \sum_{k=0}^{n-2} M_k M_{n-2-k} \,, \qquad M_0 = M_1 = 1 \,.$$

$$\begin{aligned}
&p_n(t,x,y) \\
&= \int_0^{t/2} \int_{\mathbb{R}^d} p_{n-1}(s,x,z) b(z) \cdot \nabla_z p(t-s,z,y) dz ds \\
&+ \int_{t/2}^{t} \int_{\mathbb{R}^d} \nabla_z p(s,x,z) \cdot b(z) p_{n-1}(t-s,z,y) dz ds \\
&+ \sum_{k=0}^{n-2} \int_{t/2}^{t} \int_0^{t/2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} dz dw dr ds \\
&p_{n-2-k}(r,x,w) \big[\nabla_z \cdot (b(w) \cdot \nabla_w p(s-r,w,z) b(z)) \big] p_k(t-s,z,y)
\end{aligned}$$

$$M_n = M_{n-1} + \sum_{k=0}^{n-2} M_k M_{n-2-k}\,, \qquad M_0 = M_1 = 1\,.$$

$$\begin{aligned}
&|p_n(t,x,y)| \\
&\le M_{n-1} C^{n-1} \int_0^{t/2} \int_{\mathbb{R}^d} p(s,x,z)|b(z)||\nabla_z p(t-s,z,y)|dzds \\
&+ M_{n-1} C^{n-1} \int_{t/2}^{t} \int_{\mathbb{R}^d} |\nabla_z p(s,x,z)||b(z)|p_{n-1}(t-s,z,y)dzds \\
&+ \sum_{k=0}^{n-2} M_{n-2-k} C^{n-k-2} M_k C^k \int_{t/2}^{t} \int_0^{t/2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} dzdwdrds \\
&p(r,x,w)\Big|\nabla_z \cdot (b(w) \cdot \nabla_w p(s-r,w,z)b(z))\Big|p(t-s,z,y)
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estimates of $\tilde{p}$

Theorem

There is a constant $\eta = \eta(\alpha, d, C_b)$ *such that if* $||b||_{M_1^{1-\alpha}} < \eta$, *then the function*

$$\tilde{p}(t,x,y) = \sum_{n=0}^{\infty} p_n(t,x,y)$$

is the transition density of the semigroup with the (weak) generator $\Delta^{\alpha/2} + b \cdot \nabla$. *Furthermore there is a constant* $K = K(d, \alpha, C_b, r)$ *such that*

$$K^{-1} p(t,x,y) \leq \tilde{p}(t,x,y) \leq K p(t,x,y)$$