On potential theory of subordinate Brownian motion in unbounded sets

Zoran Vondraček

(joint work with Panki Kim and Renming Song)

Department of Mathematics
University of Zagreb
Croatia

Będlewo, Sept. 10-14, 2012

Martin boundary

Let $X = (X_t, \mathbb{P}_x)$ be rotationally invariant Lévy process in $\mathbb{R}^d$, $D \subset \mathbb{R}^d$ open, X^D the killed process, $G_D(x, y)$ the Green function of X^D.

Martin boundary

Let $X = (X_t, \mathbb{P}_x)$ be rotationally invariant Lévy process in $\mathbb{R}^d$, $D \subset \mathbb{R}^d$ open, X^D the killed process, $G_D(x, y)$ the Green function of X^D.
Fix $x_0 \in D$ and define $M_D(x, y) := \frac{G_D(x,y)}{G_D(x_0,y)}$, $x, y \in D$.

Martin boundary

Let $X = (X_t, \mathbb{P}_x)$ be rotationally invariant Lévy process in $\mathbb{R}^d$, $D \subset \mathbb{R}^d$ open, X^D the killed process, $G_D(x,y)$ the Green function of X^D.

Fix $x_0 \in D$ and define $M_D(x,y) := \frac{G_D(x,y)}{G_D(x_0,y)}$, $x, y \in D$.

D has a Martin boundary $\partial_M D$ with respect to X^D satisfying the following properties:

(1) $D \cup \partial_M D$ is compact metric space;

(2) D is open and dense in $D \cup \partial_M D$, and its relative topology coincides with its original topology;

(3) $M_D(x, \cdot\,)$ can be uniquely extended to $\partial_M D$ in such a way that, $M_D(x,y)$ converges to $M_D(x,z)$ as $y \to z \in \partial_M D$, the function $x \to M_D(x,z)$ is excessive with respect to X^D, the function $(x,z) \to M_D(x,z)$ is jointly continuous on $D \times \partial_M D$ and $M_D(\cdot, z_1) \neq M_D(\cdot, z_2)$ if $z_1 \neq z_2$.

Martin boundary

Let $X = (X_t, \mathbb{P}_x)$ be rotationally invariant Lévy process in $\mathbb{R}^d$, $D \subset \mathbb{R}^d$ open, X^D the killed process, $G_D(x,y)$ the Green function of X^D.

Fix $x_0 \in D$ and define $M_D(x,y) := \frac{G_D(x,y)}{G_D(x_0,y)}$, $x, y \in D$.

D has a Martin boundary $\partial_M D$ with respect to X^D satisfying the following properties:

(1) $D \cup \partial_M D$ is compact metric space;

(2) D is open and dense in $D \cup \partial_M D$, and its relative topology coincides with its original topology;

(3) $M_D(x, \cdot\,)$ can be uniquely extended to $\partial_M D$ in such a way that, $M_D(x,y)$ converges to $M_D(x,z)$ as $y \to z \in \partial_M D$, the function $x \to M_D(x,z)$ is excessive with respect to X^D, the function $(x,z) \to M_D(x,z)$ is jointly continuous on $D \times \partial_M D$ and $M_D(\cdot, z_1) \neq M_D(\cdot, z_2)$ if $z_1 \neq z_2$.

The minimal Martin boundary of X^D is defined as

$$\partial_m D = \{z \in \partial_M D : M_D(\cdot, z) \text{ is minimal harmonic with respect to } X^D\}.$$

Some history

The notion of Martin boundary goes back to Robert S. Martin (1941) for the case classical harmonic functions (i.e. X is Brownian motion).

Some history

The notion of Martin boundary goes back to Robert S. Martin (1941) for the case classical harmonic functions (i.e. X is Brownian motion). General theory of Martin boundary for strong Markov processes (in duality) developed by Kunita and Watanabe (1965).

Some history

The notion of Martin boundary goes back to Robert S. Martin (1941) for the case classical harmonic functions (i.e. X is Brownian motion). General theory of Martin boundary for strong Markov processes (in duality) developed by Kunita and Watanabe (1965).

X is Brownian motion, D bounded Lipschitz domain D: Hunt and Wheeden (1970) proved that the (minimal) Martin boundary can be identified with the Euclidean boundary.

Some history

The notion of Martin boundary goes back to Robert S. Martin (1941) for the case classical harmonic functions (i.e. X is Brownian motion). General theory of Martin boundary for strong Markov processes (in duality) developed by Kunita and Watanabe (1965).

X is Brownian motion, D bounded Lipschitz domain D: Hunt and Wheeden (1970) proved that the (minimal) Martin boundary can be identified with the Euclidean boundary.

X rotationally invariant α-stable process, $0 < \alpha < 2$. Identification of the (minimal) Martin boundary with the Euclidean boundary:

(1) Bounded Lipschitz domain: Chen and Song (1998) and Bogdan (1999);
(2) Bounded κ-fat open set: Song and Wu (1999).

Some history

The notion of Martin boundary goes back to Robert S. Martin (1941) for the case classical harmonic functions (i.e. X is Brownian motion). General theory of Martin boundary for strong Markov processes (in duality) developed by Kunita and Watanabe (1965).

X is Brownian motion, D bounded Lipschitz domain D: Hunt and Wheeden (1970) proved that the (minimal) Martin boundary can be identified with the Euclidean boundary.
X rotationally invariant α-stable process, $0 < \alpha < 2$. Identification of the (minimal) Martin boundary with the Euclidean boundary:

(1) Bounded Lipschitz domain: Chen and Song (1998) and Bogdan (1999);

(2) Bounded κ-fat open set: Song and Wu (1999).

Certain subordinate BM, D bounded κ-fat open set: Kim, Song, V. (2009).

Martin boundary for unbounded sets?

In all mentioned results D is bounded. The reason: Proofs depend on the boundary Harnack principle for non-negative harmonic functions which implies the existence of the limit $\lim_{y \to z \in \partial D} M_D(x, y)$.

Martin boundary for unbounded sets?

In all mentioned results D is bounded. The reason: Proofs depend on the boundary Harnack principle for non-negative harmonic functions which implies the existence of the limit $\lim_{y \to z \in \partial D} M_D(x,y)$.

Results for unbounded sets. Let $H = \{x = (\tilde{x}, x_d) \in \mathbb{R}^d : x_d > 0\}$ be the upper half-space.

Martin boundary for unbounded sets?

In all mentioned results D is bounded. The reason: Proofs depend on the boundary Harnack principle for non-negative harmonic functions which implies the existence of the limit $\lim_{y\to z\in\partial D} M_D(x,y)$.

Results for unbounded sets. Let $H=\{x=(\tilde{x},x_d)\in\mathbb{R}^d : x_d>0\}$ be the upper half-space.
For $0<\alpha\le 2$, $G_H(x,y)$ known explicitly, implying that $\partial_M H=\partial_m H=\partial H\cup\{\infty\}$. The Martin kernel given by (with $x_0=(\tilde{0},1)$)

$$M_H(x,z)=\frac{x_d^{\alpha/2}}{|x-z|^d}\,(1+|z|^2)^{\alpha/2}\,,\quad M_H(x,\infty)=x_d^{\alpha/2}\,.$$

Martin boundary for unbounded sets?

In all mentioned results D is bounded. The reason: Proofs depend on the boundary Harnack principle for non-negative harmonic functions which implies the existence of the limit $\lim_{y\to z\in\partial D} M_D(x,y)$.

Results for unbounded sets. Let $H=\{x=(\tilde{x},x_d)\in\mathbb{R}^d : x_d>0\}$ be the upper half-space.
For $0<\alpha\le 2$, $G_H(x,y)$ known explicitly, implying that $\partial_M H=\partial_m H=\partial H\cup\{\infty\}$. The Martin kernel given by (with $x_0=(\tilde{0},1)$)

$$M_H(x,z)=\frac{x_d^{\alpha/2}}{|x-z|^d}\,(1+|z|^2)^{\alpha/2}\,,\quad M_H(x,\infty)=x_d^{\alpha/2}\,.$$

In case of unbounded open D, inversion through the sphere implies the existence of $M_D(x,\infty):=\lim_{|y|\to\infty,\,y\in D} M_D(x,y)$:
Bogdan, Kulczycki, Kwaśnicki (2008)

Finite part of Martin boundary

A point $z \in \partial_M H$ is called a finite Martin boundary point if there exists a bounded sequence $(y_n)_{n\geq 1}$ converging to z in the Martin topology.

Finite part of Martin boundary

A point $z \in \partial_M H$ is called a finite Martin boundary point if there exists a bounded sequence $(y_n)_{n\geq 1}$ converging to z in the Martin topology.

A point z is called an infinite Martin boundary point if every sequence $(w_n)_{n\geq 1}$ converging to w in the Martin topology is unbounded.

Finite part of Martin boundary

A point $z \in \partial_M H$ is called a finite Martin boundary point if there exists a bounded sequence $(y_n)_{n\geq 1}$ converging to z in the Martin topology.

A point z is called an infinite Martin boundary point if every sequence $(w_n)_{n\geq 1}$ converging to w in the Martin topology is unbounded.

In case X is a subordinate Brownian motion satisfying certain condition, the finite part of the Martin boundary of H can be identified with the Euclidean boundary ∂H, Kim, Song, V. (2011).

Goal of the talk

Show that $\partial_M H = \partial_m H = \partial H \cup \{\infty\}$ for a class of subordinate Brownian motions. Equivalently, there is only one infinite Martin boundary point and it corresponds to a minimal harmonic function.

Goal of the talk

Show that $\partial_M H = \partial_m H = \partial H \cup \{\infty\}$ for a class of subordinate Brownian motions. Equivalently, there is only one infinite Martin boundary point and it corresponds to a minimal harmonic function.

Remark 1: One needs assumptions on the behavior of the process for large time – large space.

Goal of the talk

Show that $\partial_M H = \partial_m H = \partial H \cup \{\infty\}$ for a class of subordinate Brownian motions. Equivalently, there is only one infinite Martin boundary point and it corresponds to a minimal harmonic function.

Remark 1: One needs assumptions on the behavior of the process for large time – large space.

Remark 2: Case $d = 1$. M. Silverstein proved in 1980 that $\partial_m(0,\infty) = \{0,\infty\}$ (two minimal harmonic functions: renewal function of the ladder height process and its density).

Goal of the talk

Show that $\partial_M H = \partial_m H = \partial H \cup \{\infty\}$ for a class of subordinate Brownian motions. Equivalently, there is only one infinite Martin boundary point and it corresponds to a minimal harmonic function.

Remark 1: One needs assumptions on the behavior of the process for large time – large space.
Remark 2: Case $d = 1$. M. Silverstein proved in 1980 that $\partial_m(0, \infty) = \{0, \infty\}$ (two minimal harmonic functions: renewal function of the ladder height process and its density). The full Martin boundary can be larger.

Subordinators

$S=(S_t)_{t\geq 0}$ a subordinator with the Laplace exponent ϕ:

$$\mathbb{E}[e^{-\lambda S_t}]=e^{-t\phi(\lambda)}\,, \qquad \phi(t)=\int_{(0,\infty)}(1-e^{-\lambda t})\,\mu(dt)$$

Subordinators

$S = (S_t)_{t\geq 0}$ a subordinator with the Laplace exponent ϕ:

$$\mathbb{E}[e^{-\lambda S_t}] = e^{-t\phi(\lambda)}, \qquad \phi(t) = \int_{(0,\infty)} (1 - e^{-\lambda t})\, \mu(dt)$$

Assumptions on ϕ: ϕ is CBF – $\mu(dt) = \mu(t)\, dt$ where μ is CM.
Consequence: the renewal measure has a CM density u. WLOG $\phi(1) = 1$.

Subordinators

$S=(S_t)_{t\ge 0}$ a subordinator with the Laplace exponent ϕ:

$$\mathbb{E}[e^{-\lambda S_t}]=e^{-t\phi(\lambda)}, \qquad \phi(t)=\int_{(0,\infty)}(1-e^{-\lambda t})\,\mu(dt)$$

Assumptions on ϕ: ϕ is CBF – $\mu(dt)=\mu(t)\,dt$ where μ is CM.
Consequence: the renewal measure has a CM density u. WLOG $\phi(1)=1$.
Upper and lower scaling conditions at infinity and at zero:
(H1): There exist constants $0<\delta_1\le\delta_2<1$ and $a_1,a_2>0$ such that

$$a_1\lambda^{\delta_1}\phi(t)\le\phi(\lambda t)\le a_2\lambda^{\delta_2}\phi(t), \quad \lambda\ge 1, t\ge 1\,.$$

Subordinators

$S=(S_t)_{t\geq 0}$ a subordinator with the Laplace exponent ϕ:

$$\mathbb{E}[e^{-\lambda S_t}] = e^{-t\phi(\lambda)}\,, \qquad \phi(t) = \int_{(0,\infty)} (1-e^{-\lambda t})\, \mu(dt)$$

Assumptions on ϕ: ϕ is CBF – $\mu(dt)=\mu(t)\,dt$ where μ is CM.
Consequence: the renewal measure has a CM density u. WLOG $\phi(1)=1$.
Upper and lower scaling conditions at infinity and at zero:
(H1): There exist constants $0<\delta_1\leq\delta_2<1$ and $a_1, a_2>0$ such that

$$a_1\lambda^{\delta_1}\phi(t)\leq\phi(\lambda t)\leq a_2\lambda^{\delta_2}\phi(t), \quad \lambda\geq 1, t\geq 1\,.$$

(H2): There exist constants $0<\delta_3\leq\delta_4<1$ and $a_3, a_4>0$ such that

$$a_3\lambda^{\delta_4}\phi(t)\leq\phi(\lambda t)\leq a_4\lambda^{\delta_3}\phi(t), \quad \lambda\leq 1, t\leq 1\,.$$

Examples

If $0 < \alpha < 2$ and $\widetilde{\ell}$ slowly varying at infinity, then

$$\phi(\lambda) \asymp \lambda^{\alpha/2}\widetilde{\ell}(\lambda)\,, \quad \lambda \to \infty\,,$$

implies (H1). Assumption on the behavior of the subordinator (hence SBM) for small time, small space.

Examples

If $0 < \alpha < 2$ and $\widetilde{\ell}$ slowly varying at infinity, then

$$\phi(\lambda) \asymp \lambda^{\alpha/2}\widetilde{\ell}(\lambda)\,, \quad \lambda \to \infty\,,$$

implies (H1). Assumption on the behavior of the subordinator (hence SBM) for small time, small space.

If $0 < \beta < 2$ and ℓ slowly varying at infinity, then

$$\phi(\lambda) \asymp \lambda^{\beta/2}\ell(\lambda)\,, \quad \lambda \to 0\,,$$

implies (H2). Assumption on the behavior of the subordinator (hence SBM) for large time, large space.

Properties of the potential and the Lévy density

There exists a constant $C = C(\phi) > 0$ such that

$$u(t) \le Ct^{-1}\phi(t^{-1})^{-1}, \qquad \mu(t) \le Ct^{-1}\phi(t^{-1}), \qquad \forall\, t \in (0,\infty).$$

(H1): $u(t) \ge C^{-1}t^{-1}\phi(t^{-1})^{-1}, \quad \mu(t) \ge C^{-1}t^{-1}\phi(t^{-1}), \quad \forall\, t \in (0,1],$

(H2): $u(t) \ge C^{-1}t^{-1}\phi(t^{-1})^{-1}, \quad \mu(t) \ge C^{-1}t^{-1}\phi(t^{-1}), \quad \forall\, t \in [1,\infty)$

Properties of the potential and the Lévy density

There exists a constant $C = C(\phi) > 0$ such that

$$u(t) \le C t^{-1} \phi(t^{-1})^{-1}, \qquad \mu(t) \le C t^{-1} \phi(t^{-1}), \qquad \forall\, t \in (0, \infty).$$

(H1): $u(t) \ge C^{-1} t^{-1} \phi(t^{-1})^{-1}, \quad \mu(t) \ge C^{-1} t^{-1} \phi(t^{-1}), \quad \forall\, t \in (0, 1],$

(H2): $u(t) \ge C^{-1} t^{-1} \phi(t^{-1})^{-1}, \quad \mu(t) \ge C^{-1} t^{-1} \phi(t^{-1}), \quad \forall\, t \in [1, \infty)$

We write

$$u(t) \asymp t^{-1} \phi(t^{-1})^{-1}, \quad \mu(t) \asymp t^{-1} \phi(t^{-1}), \qquad t \in (0, \infty).$$

Subordinate Brownian motion

$W = (W_t, \mathbb{P}_x)$ d-dimensional Brownian motion, $S = (S_t)$ and independent subordinator with the Laplace exponent ϕ satisfying (H1), (H2) and CBF.

Subordinate Brownian motion

$W = (W_t, \mathbb{P}_x)$ d-dimensional Brownian motion, $S = (S_t)$ and independent subordinator with the Laplace exponent ϕ satisfying (H1), (H2) and CBF. The SBM is the process $X = (X_t)_{t \geq 0}$ defined as $X_t := W_{S_t}$.

Subordinate Brownian motion

$W=(W_t,\mathbb{P}_x)$ d-dimensional Brownian motion, $S=(S_t)$ and independent subordinator with the Laplace exponent ϕ satisfying (H1), (H2) and CBF. The SBM is the process $X=(X_t)_{t\geq 0}$ defined as $X_t := W_{S_t}$.

X is a Lévy process with characteristic exponent $\Phi(x)=\phi(|x|^2)$ and Lévy measure with density $J(x)=j(|x|)$ where

$$j(r)=\int_0^\infty (4\pi t)^{-d/2} e^{-r^2/4t}\,\mu(t)\,dt\,,\quad r>0\,.$$

Subordinate Brownian motion

$W=(W_t,\mathbb{P}_x)$ d-dimensional Brownian motion, $S=(S_t)$ and independent subordinator with the Laplace exponent ϕ satisfying (H1), (H2) and CBF. The SBM is the process $X=(X_t)_{t\geq 0}$ defined as $X_t := W_{S_t}$.

X is a Lévy process with characteristic exponent $\Phi(x)=\phi(|x|^2)$ and Lévy measure with density $J(x)=j(|x|)$ where

$$j(r)=\int_0^\infty (4\pi t)^{-d/2}e^{-r^2/4t}\,\mu(t)\,dt\,,\quad r>0\,.$$

Assume X is transient ($\Longleftrightarrow \int_0^1 \phi(\lambda)^{-1}\lambda^{d/2-1}\,d\lambda<\infty$); then X has the Green function $G(x,y)=G(x-y)=g(|x-y|)$ where

$$g(r)=\int_0^\infty (4\pi t)^{-d/2}e^{-r^2/4t}\,u(t)\,dt\,,\quad r>0\,.$$

Renewal measure of the ladder height process

Let W be the one-dimensional Brownian motion. Then $X_t := W_{S_t}$ is one-dimensional Lévy process. The ladder height process $H = (H_t)_{t\geq 0}$ is a subordinator.

Renewal measure of the ladder height process

Let W be the one-dimensional Brownian motion. Then $X_t := W_{S_t}$ is one-dimensional Lévy process. The ladder height process $H = (H_t)_{t\geq 0}$ is a subordinator. The renewal measure of H is defined by

$$V(A) := \mathbb{E} \int_0^\infty 1_{(H_t \in A)}\, dt\,,$$

and $V(t) := V([0,t])$ is the renewal function.

Renewal measure of the ladder height process

Let W be the one-dimensional Brownian motion. Then $X_t := W_{S_t}$ is one-dimensional Lévy process. The ladder height process $H = (H_t)_{t\geq 0}$ is a subordinator. The renewal measure of H is defined by

$$V(A) := \mathbb{E} \int_0^\infty 1_{(H_t \in A)}\, dt\,,$$

and $V(t) := V([0,t])$ is the renewal function.

It holds that

$$V(t) \asymp \phi(r^{-2})^{-1}\,, \quad \text{for all } r > 0\,.$$

Green function and Lévy measure

Theorem: Assume (H1) and (H2).

Green function and Lévy measure

Theorem: Assume (H1) and (H2). (a) Then

$$J(x) \asymp |x|^{-d}\phi(|x|^{-2}), \quad x \neq 0 .$$

Green function and Lévy measure

Theorem: Assume (H1) and (H2). (a) Then

$$J(x) \asymp |x|^{-d}\phi(|x|^{-2}), \quad x \neq 0.$$

(b) X is transient and if $d > 2(\delta_2 \vee \delta_4)$, then

$$G(x) \asymp |x|^{-d}\phi(|x|^{-2})^{-1}, \quad x \neq 0.$$

Green function and Lévy measure

Theorem: Assume (H1) and (H2). (a) Then

$$J(x) \asymp |x|^{-d}\phi(|x|^{-2}), \quad x \neq 0.$$

(b) X is transient and if $d > 2(\delta_2 \vee \delta_4)$, then

$$G(x) \asymp |x|^{-d}\phi(|x|^{-2})^{-1}, \quad x \neq 0.$$

Corollary: (Doubling property) $J(2x) \asymp J(x)$, $x \neq 0$.

Scaling properties

For $a > 0$, let $\phi^a(\lambda) := \phi(\lambda a^{-2})/\phi(a^{-2})$, S^a corresponding subordinator, $X^a = X_{S^a_t}$ SBM.

Scaling properties

For $a>0$, let $\phi^a(\lambda):=\phi(\lambda a^{-2})/\phi(a^{-2})$, S^a corresponding subordinator, $X^a=X_{S^a_t}$ SBM.
$\{X^a_t\}_{t\geq 0}\overset{\text{law}}{=}\{a^{-1}X_{t/\phi(a^{-2})}\}_{t\geq 0}$.

$$J^a(x)\asymp |x|^{-d}\phi^a(|x|^{-2}), \quad G^a(x)\asymp |x|^{-d}\phi^a(|x|^{-2})^{-1}$$

(constant independent of a).

Scaling properties

For $a > 0$, let $\phi^a(\lambda) := \phi(\lambda a^{-2})/\phi(a^{-2})$, S^a corresponding subordinator, $X^a = X_{S_t^a}$ SBM.
$\{X_t^a\}_{t\geq 0} \stackrel{\text{law}}{=} \{a^{-1}X_{t/\phi(a^{-2})}\}_{t\geq 0}$.

$$J^a(x) \asymp |x|^{-d}\phi^a(|x|^{-2}), \quad G^a(x) \asymp |x|^{-d}\phi^a(|x|^{-2})^{-1}$$

(constant independent of a).

Define $\Phi^a(r) := \phi^a(r^{-2})^{-1}$, $r > 0$. Then

$$a_5 \left(\frac{R}{r}\right)^{2(\delta_1\wedge\delta_3)} \leq \frac{\Phi^a(R)}{\Phi^a(r)} \leq a_6 \left(\frac{R}{r}\right)^{2(\delta_2\vee\delta_4)} \qquad a > 0,\ 0 < r < R < \infty\,.$$

Scaling properties

For $a > 0$, let $\phi^a(\lambda) := \phi(\lambda a^{-2})/\phi(a^{-2})$, S^a corresponding subordinator, $X^a = X_{S_t^a}$ SBM.
$\{X_t^a\}_{t\geq 0} \stackrel{\text{law}}{=} \{a^{-1}X_{t/\phi(a^{-2})}\}_{t\geq 0}$.

$$J^a(x) \asymp |x|^{-d}\phi^a(|x|^{-2}), \quad G^a(x) \asymp |x|^{-d}\phi^a(|x|^{-2})^{-1}$$

(constant independent of a).

Define $\Phi^a(r) := \phi^a(r^{-2})^{-1}$, $r > 0$. Then

$$a_5 \left(\frac{R}{r}\right)^{2(\delta_1\wedge\delta_3)} \leq \frac{\Phi^a(R)}{\Phi^a(r)} \leq a_6 \left(\frac{R}{r}\right)^{2(\delta_2\vee\delta_4)} \qquad a > 0,\ 0 < r < R < \infty\,.$$

X^a satisfies conditions of Chen-Kumagai, PTRF (2008)

Uniform BHP

Recall that $u : \mathbb{R}^d \to [0, \infty)$ is regular harmonic in open $D \subset \mathbb{R}^d$ with respect to X if

$$u(x) = \mathbb{E}_x \left[u(X_{\tau_D}) : \tau_D < \infty \right] , \quad \text{for all } x \in D .$$

Uniform BHP

Recall that $u : \mathbb{R}^d \to [0, \infty)$ is regular harmonic in open $D \subset \mathbb{R}^d$ with respect to X if

$$u(x) = \mathbb{E}_x \left[u(X_{\tau_D}) : \tau_D < \infty \right] , \quad \text{for all } x \in D .$$

Theorem: There exists a constant $c = c(\phi, d) > 0$ such that for every $z_0 \in \mathbb{R}^d$, every open set $D \subset \mathbb{R}^d$, every $r > 0$ and for any nonnegative functions u, v in $\mathbb{R}^d$ which are regular harmonic in $D \cap B(z_0, r)$ with respect to X and vanish in $D^c \cap B(z_0, r)$, we have

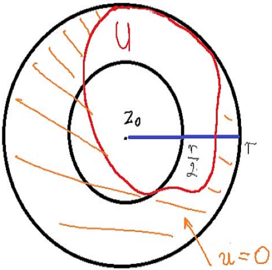
$$\frac{u(x)}{v(x)} \le c \, \frac{u(y)}{v(y)} \quad \text{for all } x, y \in D \cap B(z_0, r/2) .$$

Lemma: For every $z_0 \in \mathbb{R}^d$, every open set $U \subset B(z_0, r)$ and for any nonnegative function u in $\mathbb{R}^d$ which is regular harmonic in U with respect to X and vanishes a.e. in $U^c \cap B(z_0, r)$ it holds that

$$u(x) \asymp \mathbb{E}_x[\tau_U] \int_{B(z_0, r/2)^c} j(|y - z_0|) u(y) dy \,, \quad x \in U \cap B(z_0, r/2) \,.$$

Lemma: For every $z_0 \in \mathbb{R}^d$, every open set $U \subset B(z_0, r)$ and for any nonnegative function u in $\mathbb{R}^d$ which is regular harmonic in U with respect to X and vanishes a.e. in $U^c \cap B(z_0, r)$ it holds that

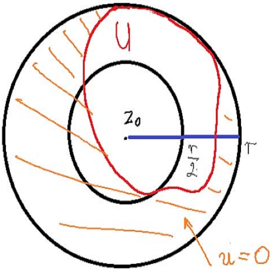
$$u(x) \asymp \mathbb{E}_x[\tau_U] \int_{B(z_0, r/2)^c} j(|y - z_0|) u(y) dy \,, \quad x \in U \cap B(z_0, r/2) \,.$$



Lemma: For every $z_0 \in \mathbb{R}^d$, every open set $U \subset B(z_0, r)$ and for any nonnegative function u in $\mathbb{R}^d$ which is regular harmonic in U with respect to X and vanishes a.e. in $U^c \cap B(z_0, r)$ it holds that

$$u(x) \asymp \mathbb{E}_x[\tau_U] \int_{B(z_0, r/2)^c} j(|y - z_0|) u(y) dy\,, \quad x \in U \cap B(z_0, r/2)\,.$$

For all $r \in (0, 1]$ under (H1) (Kim, Song, V. (2011)), for all $r \in (0, \infty)$ under (H1) and (H2).



Take $z_0 = 0$. Then the above reads:

$$u(x) \asymp \int_U G_U(x,y)\,dy \int_{B(0,r/2)^c} j(|y|)u(y)dy\,, \quad x \in U \cap B(0,r/2)\,.$$

Take $z_0 = 0$. Then the above reads:

$$u(x) \asymp \int_U G_U(x,y)\, dy \int_{B(0,r/2)^c} j(|y|)u(y)dy\,, \quad x \in U \cap B(0,r/2)\,.$$

In case of rotationally invariant α-stable process, M. Kwaśnicki (2009) used the inversion through the sphere $B(0,\sqrt{r})$ to obtain a BHP at infinity.

Boundary Harnack inequality at infinity

Recall that the Poisson kernel $K_U(x,z)$ is the exit density from an open set U: $\mathbb{P}_x(X_{\tau_U} \in B) = \int_B K_U(x,z)\,dy\,, \quad B \subset \overline{U}^c\,,$

$$K_U(x,z) = \int_U G_U(x,y) j(|y-z|)\,dy\,, \quad x \in U, z \in \overline{U}^c\,.$$

Boundary Harnack inequality at infinity

Recall that the Poisson kernel $K_U(x,z)$ is the exit density from an open set U: $\mathbb{P}_x(X_{\tau_U} \in B) = \int_B K_U(x,z)\, dy\,, \quad B \subset \overline{U}^c\,,$

$$K_U(x,z) = \int_U G_U(x,y) j(|y-z|)\, dy\,, \quad x \in U, z \in \overline{U}^c\,.$$

If u regular harmonic in U, then $u(x) = \int_{\overline{U}^c} K_U(x,z) u(z)\, dz$.

Boundary Harnack inequality at infinity

Recall that the Poisson kernel $K_U(x,z)$ is the exit density from an open set U: $\mathbb{P}_x(X_{\tau_U} \in B) = \int_B K_U(x,z)\,dy\,, \quad B \subset \overline{U}^c\,,$

$$K_U(x,z) = \int_U G_U(x,y) j(|y-z|)\,dy\,, \quad x \in U, z \in \overline{U}^c\,.$$

If u regular harmonic in U, then $u(x) = \int_{\overline{U}^c} K_U(x,z) u(z)\,dz$.

Additional technical assumption:

$$\text{(A):} \qquad 2\delta_2 - \delta_1 < 1 \quad \text{and} \quad 2\delta_4 - \delta_3 < 1\,.$$

BHP at infinity – continuation

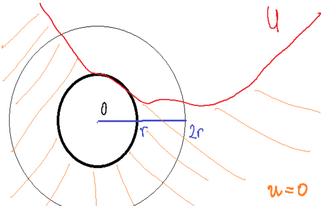
Theorem: There exists $C = C(\phi) > 1$ such that for all $r \geq 1$, for all open sets $U \subset \overline{B}(0, r)^c$ and all nonnegative functions u on $\mathbb{R}^d$ that are regular harmonic in U and vanish on $\overline{B}(0, r)^c \setminus U$, it holds that

$$\frac{1}{C} \leq \frac{u(x)}{K_U(x, 0) \int_{B(0,2r)} u(z)\, dz} \leq C \,, \qquad \text{for all } x \in U \cap \overline{B}(0, 2r)^c \,.$$

BHP at infinity – continuation

Theorem: There exists $C = C(\phi) > 1$ such that for all $r \geq 1$, for all open sets $U \subset \overline{B}(0,r)^c$ and all nonnegative functions u on $\mathbb{R}^d$ that are regular harmonic in U and vanish on $\overline{B}(0,r)^c \setminus U$, it holds that

$$\frac{1}{C} \leq \frac{u(x)}{K_U(x,0) \int_{B(0,2r)} u(z)\, dz} \leq C\,, \qquad \text{for all } x \in U \cap \overline{B}(0,2r)^c\,.$$

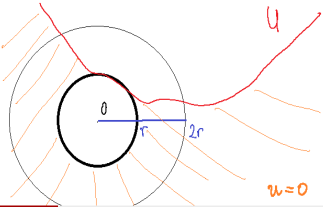


BHP at infinity – continuation

Theorem: There exists $C = C(\phi) > 1$ such that for all $r \geq 1$, for all open sets $U \subset \overline{B}(0,r)^c$ and all nonnegative functions u on $\mathbb{R}^d$ that are regular harmonic in U and vanish on $\overline{B}(0,r)^c \setminus U$, it holds that

$$\frac{1}{C} \leq \frac{u(x)}{K_U(x,0)\int_{B(0,2r)} u(z)\,dz} \leq C\,, \qquad \text{for all } x \in U \cap \overline{B}(0,2r)^c\,.$$

$$u(x) \asymp \int_U G_U(x,y) j(|y|)\,dy \int_{B(0,2r)} u(z)\,dz\,, \quad x \in U \cap \overline{B}(0,2r)^c\,.$$



Corollaries

Corollary: There exists $C = C(\phi) > 1$ such that for all $r \geq 1$, for all open sets $U \subset \overline{B}(0,r)^c$ and all nonnegative functions u and v on $\mathbb{R}^d$ that are regular harmonic in U and vanish on $\overline{B}(0,r)^c \setminus U$, it holds that

$$C^{-1}\frac{u(y)}{v(y)} \leq \frac{u(x)}{v(x)} \leq C\frac{u(y)}{v(y)}, \qquad \text{for all } x, y \in U \cap \overline{B}(0,2r)^c.$$

Corollaries

Corollary: There exists $C = C(\phi) > 1$ such that for all $r \geq 1$, for all open sets $U \subset \overline{B}(0,r)^c$ and all nonnegative functions u and v on $\mathbb{R}^d$ that are regular harmonic in U and vanish on $\overline{B}(0,r)^c \setminus U$, it holds that

$$C^{-1}\frac{u(y)}{v(y)} \leq \frac{u(x)}{v(x)} \leq C\frac{u(y)}{v(y)}, \qquad \text{for all } x, y \in U \cap \overline{B}(0,2r)^c .$$

Corollary: Let $r \geq 1$ and $U \subset \overline{B}(0,r)^c$. If u is a non-negative function on $\mathbb{R}^d$ which is regular harmonic in U and vanishes on $\overline{B}(0,r)^c \setminus U$, then

$$\lim_{|x|\to\infty} u(x) = 0 .$$

Corollaries

Corollary: There exists $C = C(\phi) > 1$ such that for all $r \geq 1$, for all open sets $U \subset \overline{B}(0,r)^c$ and all nonnegative functions u and v on $\mathbb{R}^d$ that are regular harmonic in U and vanish on $\overline{B}(0,r)^c \setminus U$, it holds that

$$C^{-1}\frac{u(y)}{v(y)} \leq \frac{u(x)}{v(x)} \leq C\frac{u(y)}{v(y)}, \qquad \text{for all } x, y \in U \cap \overline{B}(0,2r)^c\,.$$

Corollary: Let $r \geq 1$ and $U \subset \overline{B}(0,r)^c$. If u is a non-negative function on $\mathbb{R}^d$ which is regular harmonic in U and vanishes on $\overline{B}(0,r)^c \setminus U$, then

$$\lim_{|x|\to\infty} u(x) = 0\,.$$

Remark: Not true if *regular harmonic* is replaced by *harmonic*: $w(x) = w(\widetilde{x}, x_d) := V((x_d)^+)$ is harmonic in the upper half-space $H \subset B((\widetilde{0}, -1), 1)^c$, vanishes on $\overline{B}((\widetilde{0}, -1), 1)^c \setminus H$, but $\lim_{x_d\to\infty} w(x) = \infty$.

Ingredients of the proof

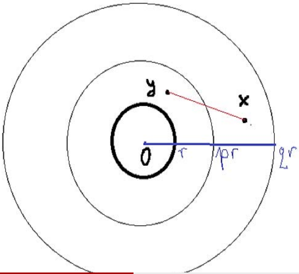
Upper bound on the Green function $\overline{B}(0,r)^c$, $r \geq 1$: Let $1 < p < q < 4$ and $b > 0$. There exist a constant $C = C(\phi, p, q, b) > 0$ such that for all $r \geq 1$, all $x \in A(0, pr, qr)$ and all $y \in \overline{B}(0,r)^c$ such that $\delta_{\overline{B}(0,r)^c}(y) < r$ and $br < |x-y|$ it holds that

$$G_{\overline{B}(0,r)^c}(x,y) \leq C \frac{V(\delta_{\overline{B}(0,r)^c}(x))}{V(|x-y|)} \frac{V(\delta_{\overline{B}(0,r)^c}(y))}{V(|x-y|)} G(x,y) \, .$$

Ingredients of the proof

Upper bound on the Green function $\overline{B}(0,r)^c$, $r \geq 1$: Let $1 < p < q < 4$ and $b > 0$. There exist a constant $C = C(\phi, p, q, b) > 0$ such that for all $r \geq 1$, all $x \in A(0, pr, qr)$ and all $y \in \overline{B}(0,r)^c$ such that $\delta_{\overline{B}(0,r)^c}(y) < r$ and $br < |x - y|$ it holds that

$$G_{\overline{B}(0,r)^c}(x,y) \leq C \frac{V(\delta_{\overline{B}(0,r)^c}(x))}{V(|x-y|)} \frac{V(\delta_{\overline{B}(0,r)^c}(y))}{V(|x-y|)} G(x,y) .$$



Ingredients of the proof – continuation

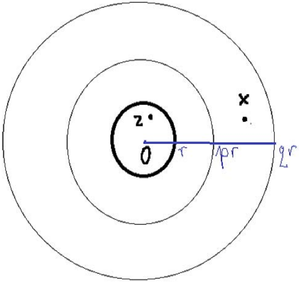
Upper bound for the Poisson kernel of $\overline{B}(0,r)^c$, $r \geq 1$: Let $1 < p < q < 4$. There exists $C = C(\phi, p, q) > 1$ such that for all $r \geq 1$, all $x \in A(0, pr, qr)$ and $z \in B(0,r)$ it holds that

$$K_{\overline{B}(0,r)^c}(x,z) \leq C\left(|x-z|^{-d}\left(\phi(r^{-2})^{-1/2}\phi((r-|z|)^{-2})^{1/2}+1\right)+r^{-d}\right).$$

Ingredients of the proof – continuation

Upper bound for the Poisson kernel of $\overline{B}(0,r)^c$, $r \geq 1$: Let $1 < p < q < 4$. There exists $C = C(\phi, p, q) > 1$ such that for all $r \geq 1$, all $x \in A(0, pr, qr)$ and $z \in B(0, r)$ it holds that

$$K_{\overline{B}(0,r)^c}(x,z) \leq C\left(|x-z|^{-d}\left(\phi(r^{-2})^{-1/2}\phi((r-|z|)^{-2})^{1/2}+1\right)+r^{-d}\right).$$



Ingredients of the proof – continuation

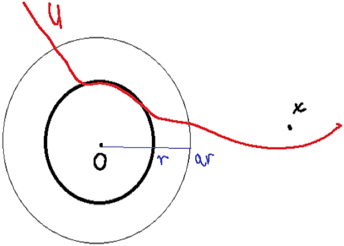
Exit probability estimate: For every $a \in (1, \infty)$, there exists a positive constant $C = C(\phi, a) > 0$ such that for any $r \in (0, \infty)$ and any open set $U \subset \overline{B}(0, r)^c$ we have

$$\mathbb{P}_x \left(X_{\tau_U} \in \overline{B(0, r)} \right) \le C r^d K_U(x, 0) , \qquad x \in U \cap \overline{B}(0, ar)^c .$$

Ingredients of the proof – continuation

Exit probability estimate: For every $a \in (1, \infty)$, there exists a positive constant $C = C(\phi, a) > 0$ such that for any $r \in (0, \infty)$ and any open set $U \subset \overline{B}(0, r)^c$ we have

$$\mathbb{P}_x \left(X_{\tau_U} \in \overline{B(0,r)} \right) \le C r^d K_U(x, 0)\,, \qquad x \in U \cap \overline{B}(0, ar)^c\,.$$



Ingredients of the proof – continuation

Ingredients of the proof – continuation

Regularization of the Poisson kernel in the spirit of Bogdan, Kulczycki and Kwaśnicki (2008) leading to

$$K_U(x,z) \asymp K_U(x,0)\left(\int_{U\cap B(0,2r)} K_U(y,z)\,dy + 1\right).$$

Oscillation reduction

Recall that $H = \{x = (\tilde{x}, x_d) : \ x_d > 0\}$ is the upper half-space, $M_H(x, y) = \frac{G_H(x,y)}{G_H(x_0,y)}$ where $x_0 = (\tilde{0}, 1)$. For any $r > 0$ let $A_r := (\tilde{0}, 2r)$.

Oscillation reduction

Recall that $H=\{x=(\tilde{x},x_d): x_d>0\}$ is the upper half-space, $M_H(x,y)=\frac{G_H(x,y)}{G_H(x_0,y)}$ where $x_0=(\tilde{0},1)$. For any $r>0$ let $A_r:=(\tilde{0},2r)$.

Lemma: For $r>0$ and $k=1,2,\dots$, let $B_k=B(0,4^k r)$. There exist $c_1=c_1(\phi,d)>0$ and $c_2=c_2(\phi,d)\in(0,1)$ such that for any $r>1$ and any non-negative function h which is regular harmonic in $H\cap\overline{B}(0,4r)^c$ and vanishes in $H^c\cap\overline{B}(0,4r)^c$ we have

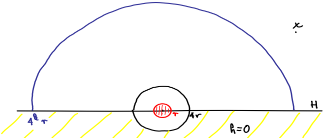
$$\mathbb{E}_x\left[h(X_{\tau_{H\cap\overline{B}_k^c}}): X_{\tau_{H\cap\overline{B}_k^c}}\in B(0,r)\right]\le c_1c_2^k h(x), \quad x\in H\cap\overline{B}_k^c, k=1,2,\dots$$

Oscillation reduction

Recall that $H=\{x=(\tilde{x},x_d):\ x_d>0\}$ is the upper half-space, $M_H(x,y)=\frac{G_H(x,y)}{G_H(x_0,y)}$ where $x_0=(\tilde{0},1)$. For any $r>0$ let $A_r:=(\tilde{0},2r)$.

Lemma: For $r>0$ and $k=1,2,\dots$, let $B_k=B(0,4^k r)$. There exist $c_1=c_1(\phi,d)>0$ and $c_2=c_2(\phi,d)\in(0,1)$ such that for any $r>1$ and any non-negative function h which is regular harmonic in $H\cap\overline{B}(0,4r)^c$ and vanishes in $H^c\cap\overline{B}(0,4r)^c$ we have

$$\mathbb{E}_x\left[h(X_{\tau_{H\cap\overline{B}_k^c}}):\ X_{\tau_{H\cap\overline{B}_k^c}}\in B(0,r)\right]\le c_1 c_2^k h(x)\,,\quad x\in H\cap\overline{B}_k^c\,,k=1,2,\dots$$



Oscillation reduction – continuation

Lemma: There exist $C = C(\phi, d) > 0$ and $\nu = \nu(d, \phi) > 0$ such that for all $r \geq 1$ and all non-negative functions u and v on $\mathbb{R}^d$ which are regular harmonic in $H \cap \overline{B}(0, r/2)^c$, vanish in $H^c \cap \overline{B}(0, r/2)^c$ and satisfy $u(A_r) = v(A_r)$, there exists the limit

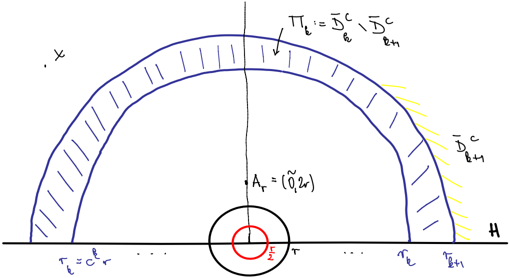
$$g = \lim_{|x| \to \infty,\, x \in H} \frac{u(x)}{v(x)},$$

and we have

$$\left| \frac{u(x)}{v(x)} - g \right| \leq C \left(\frac{|x|}{r} \right)^{-\nu}, \quad x \in H \cap \overline{B}(0, r)^c .$$

The proof is analogous to the proof in Bogdan (1997) Lemma 16. Instead of shrinking the balls to the boundary point, we use larger and larger balls, and apply BHP at infinity.

The proof is analogous to the proof in Bogdan (1997) Lemma 16. Instead of shrinking the balls to the boundary point, we use larger and larger balls, and apply BHP at infinity.



Martin kernel at infinity

Theorem: For each $x \in H$ there exists the limit

$$M_H(x,\infty) := \lim_{y\in H,\ |y|\to\infty} M_H(x,y)\,.$$

Martin kernel at infinity

Theorem: For each $x \in H$ there exists the limit

$$M_H(x,\infty) := \lim_{y\in H,\ |y|\to\infty} M_H(x,y)\,.$$

This immediately implies that the every infinite Martin boundary point can be identified with $\{\infty\}$. Since Martin kernels for different Martin boundary points are different, this gives that the infinite part of the Martin boundary is exactly $\{\infty\}$.

Minimality

Theorem: The function $M_H(\cdot, \infty)$ is harmonic in H with respect to X.

Minimality

Theorem: The function $M_H(\cdot, \infty)$ is harmonic in H with respect to X.

Lemma: (Kim, Song, V. (2011)) For every $R > 0$ and $\epsilon \in (0, 1)$, there exists a constant $c(R, \epsilon) = c(d, \phi, R, \epsilon) > 0$ such that for all $z \in \partial H$ with $|z| < R$, all $x \in H \cap B(z, \epsilon)$, and all $w \in \partial_M H \setminus (\partial H \cap B(z, 2\epsilon))$ it holds that

$$c^{-1} V(\delta_H(x)) \leq M_H(x, w) \leq cV(\delta_H(x)) .$$

Minimality

Theorem: The function $M_H(\cdot, \infty)$ is harmonic in H with respect to X.

Lemma: (Kim, Song, V. (2011)) For every $R > 0$ and $\epsilon \in (0,1)$, there exists a constant $c(R,\epsilon) = c(d,\phi,R,\epsilon) > 0$ such that for all $z \in \partial H$ with $|z| < R$, all $x \in H \cap B(z,\epsilon)$, and all $w \in \partial_M H \setminus (\partial H \cap B(z, 2\epsilon))$ it holds that

$$c^{-1} V(\delta_H(x)) \le M_H(x,w) \le cV(\delta_H(x)) .$$

Consequently, $\lim_{x \to z} M_H(x,\infty) = 0$ for every $z \in \partial H$.

Minimality

Theorem: The function $M_H(\cdot,\infty)$ is harmonic in H with respect to X.

Lemma: (Kim, Song, V. (2011)) For every $R>0$ and $\epsilon\in(0,1)$, there exists a constant $c(R,\epsilon)=c(d,\phi,R,\epsilon)>0$ such that for all $z\in\partial H$ with $|z|<R$, all $x\in H\cap B(z,\epsilon)$, and all $w\in\partial_M H\setminus(\partial H\cap B(z,2\epsilon))$ it holds that

$$c^{-1}V(\delta_H(x))\le M_H(x,w)\le cV(\delta_H(x)).$$

Consequently, $\lim_{x\to z}M_H(x,\infty)=0$ for every $z\in\partial H$.

Theorem: The function $M_H(\cdot,\infty)$ is a minimal harmonic function.

Minimality

Theorem: The function $M_H(\cdot, \infty)$ is harmonic in H with respect to X.

Lemma: (Kim, Song, V. (2011)) For every $R > 0$ and $\epsilon \in (0,1)$, there exists a constant $c(R,\epsilon) = c(d,\phi,R,\epsilon) > 0$ such that for all $z \in \partial H$ with $|z| < R$, all $x \in H \cap B(z,\epsilon)$, and all $w \in \partial_M H \setminus (\partial H \cap B(z, 2\epsilon))$ it holds that

$$c^{-1} V(\delta_H(x)) \leq M_H(x,w) \leq cV(\delta_H(x)) .$$

Consequently, $\lim_{x\to z} M_H(x,\infty) = 0$ for every $z \in \partial H$.

Theorem: The function $M_H(\cdot, \infty)$ is a minimal harmonic function. Consequently, $M_H(x,\infty) = w(x) = V((x_d)^+)$.