The topological entropy of Banach spaces

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joint work with H. Bruin

S. Banach, *Théorie des opérations linéaires*, PWN, Warsaw, 1932.

$C([0,1])$...the set of all continuous functions $f: [0,1] \to \mathbb{R}$ equipped with the supremum norm

Theorem (Banach-Mazur)

The Banach space $C([0,1])$ is universal, i.e., every real, separable Banach space X is isometrically isomorphic to a closed subspace of $C([0,1])$.

B. Levine, D. Milman, *On linear sets in space C consisting of functions of bounded variation*, Comm. Inst. Sci. Math. Méc. Univ. Kharkoff **16** (1940), 102-105.

Theorem

Every infinite-dimensional closed subspace of $C([0,1])$ *must contain a function with infinite variation.*

V.I. Gurariy, *Subspaces of differentiable functions in the space of continuous functions*, Teor. Funktsii Funktsional Anal. i Prilozhen. **4**(1967), 121-161.

Theorem

Every infinite-dimensional closed subspace E of $C([0,1])$ must contain a function which is not differentiable at some point of $[0,1]$.

P. P. Petrushev, S.L. Troyanski, *On the Banach Mazur theorem on the universality of $C([0,1])$*, C. R. Acad. Bulgare Sci. **37** (1984), 283-285.

Theorem

Every isometrically isomorphic copy of ℓ_1 in $C([0,1])$ contains a function which is non-differentiable at every point of a perfect subset of $[0,1]$.

L. Rodríguez-Piazza, *Every separable Banach space is isometric to a space of continuous nowhere differentiable functions*, Proc. Amer. Math. Soc. **123** (1995), 3649-3654.

Theorem

Every separable Banach space is isometrically isomorphic to a space of continuous nowhere differentiable functions.

S. Hencl, *Isometrical embeddings of separable Banach spaces into the set of nowhere approximately differentiable and nowhere Hölder functions*, Proc. Amer. Math. Soc. **128** (2000), 3505-3511.

Theorem

Every separable Banach space is isometrically isomorphic to a space of continuous nowhere approximately differentiable and nowhere Hölder functions.

J. B., H. Bruin, *The topological entropy of Banach spaces*, Journal of Difference Equations and Applications 18(4)(2012), 569-578.

Let $C_b(X)$ denote the set of all *bounded* continuous functions $f: X \to \mathbb{R}$ equipped with the supremum norm. Clearly, $C_b(\mathbb{R})$ is a non-separable Banach space. Let $[a, b]$ be a closed finite subinterval of $\mathbb{R}$. We identify $f: [a, b] \to \mathbb{R}$ with its extension

$$(Exf)(x) = \begin{cases} f(x) & \text{if } x \in [a, b]; \\ f(b) & \text{if } x \geq b; \\ f(a) & \text{if } x \leq a. \end{cases}$$

Under this identification, $C([a, b]) \subset C_b(\mathbb{R})$. We will deal with the topological entropy of maps from $C_b(\mathbb{R})$ defined as $h_{top}(f) := h_{top}(f|_{\overline{f(\mathbb{R})}})$.

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In the next lemma we denote by $F(X)$ a linear space of functions $f\colon\ X \to \mathbb{R}$.

Lemma

Given n linearly independent functions in $F(X)$, there exist n points $x_1, \dots, x_n \in X$ such that the vectors

$$\begin{pmatrix} f_1(x_1) \\ f_1(x_2) \\ \vdots \\ f_1(x_n) \end{pmatrix}, \begin{pmatrix} f_2(x_1) \\ f_2(x_2) \\ \vdots \\ f_2(x_n) \end{pmatrix}, \dots, \begin{pmatrix} f_n(x_1) \\ f_n(x_2) \\ \vdots \\ f_n(x_n) \end{pmatrix}$$

are linearly independent in $\mathbb{R}^n$.

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Definition

For a given set $\mathcal{B} \subset C_b(\mathbb{R})$, let

$$\begin{aligned} h^+_{top}(\mathcal{B}) &= \sup\{h_{top}(f) : f \in \mathcal{B}\}, \\ h^-_{top}(\mathcal{B}) &= \inf\{h_{top}(f) : f \in \mathcal{B},\ f \text{ is non-zero}\}. \end{aligned}$$

Proposition

If a linear space $\mathcal{B} \subset C_b(\mathbb{R})$ has dimension n, then

$$h^+_{top}(\mathcal{B}) \geq \log(n-1).$$

In particular, $h^+_{top}(\mathcal{B}) = \infty$ if $\dim(\mathcal{B}) = \infty$.

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Example

There exists an isometrically isomorphic copy E of c_0 in $C([0,1])$ such that every $f \in E$ has a finite topological entropy.

Example

There is a universal Banach space $\mathcal{A} \subset C([-1,1])$ such that $h_{top}(f) = \infty$ for every non-zero f from $\mathcal{A}$.

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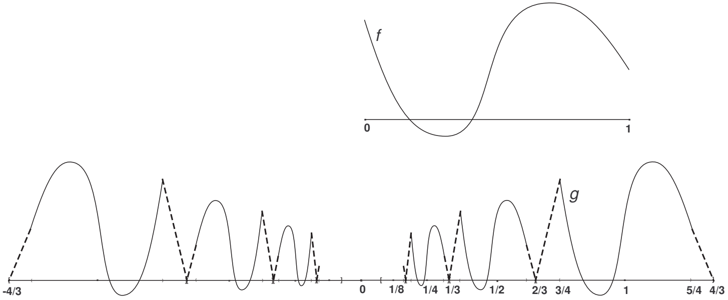

Figure: The maps $f \in C([0,1])$ and $\Psi(f) = g \in C([-\frac{4}{3}, \frac{4}{3}])$, $p_n = (\frac{1}{2})^n$, $q_n = (\frac{2}{3})^n$, $n \geq 0$.

Remark

Recall that $f \in C^\alpha(\mathbb{R})$ (f is α-*Hölder on* $\mathbb{R}$) for some $\alpha \in (0,1)$ if

$$\sup\left\{\frac{|f(x)-f(y)|}{|x-y|^\alpha}:\ x,y \in \mathbb{R},\ 0<|x-y|\le 1\right\} < \infty.$$

For some fixed $\alpha \in (0,1)$, if we choose $q_n = p_n^\alpha$ and $f \in C^\alpha([0,1])$, then $\Psi(f)$ is α-Hölder on $\mathbb{R}$. Therefore $\mathcal{A}^\alpha := \Psi(C^\alpha([0,1])) \subset C_b^\alpha(\mathbb{R})$ is a normed (infinite dimensional) linear space such that $h_{top}(f) = \infty$ for every non-zero f from $\mathcal{A}^\alpha$.

Theorem

Let $\mathcal{A} \subset C([0,1])$ be isometrically isomorphic to ℓ_1. Then $\mathcal{A}$ contains a function with infinite topological entropy.

The following statement shows that the entropy can behave extremely rigidly on a one-dimensional subspace of $C_b(\mathbb{R})$.

Theorem

For any $t \in [0, \infty]$, there exists a function $f \in C_b(\mathbb{R})$ such that for $\mathcal{B} = \{\lambda f\}_{\lambda \in \mathbb{R}}$ satisfies $h^-_{top}(\mathcal{B}) = h^+_{top}(\mathcal{B}) = t$.