Set of numbers uniformly well-approximated in the sense of Dirichlet

Lingmin LIAO
(joint work with Dong Han Kim)

Université Paris-Est Créteil (University Paris 12)

Ergodic Methods in Dynamics
Będlewo, April 2012

I. Dirichlet

Uniform Dirichlet Theorem (1842) :
Let θ, Q be real numbers with $Q \geq 1$. There exists an integer n with $1 \leq n \leq Q$, such that

$$\|n\theta\| < Q^{-1}.$$

In other words,

$$\left\{\theta : \ \forall Q > 1, \ \|n\theta\| < Q^{-1} \text{ admits a solution } 1 \leq n \leq Q\right\} = \mathbb{R}.$$

Asymptotic Dirichlet Theorem :
For any real θ, there exist infinitely many integers n such that

$$\|n\theta\| < n^{-1}.$$

In other words,

$$\left\{\theta : \ \|n\theta\| < n^{-1} \text{ for infinitely many } \ n\right\} = \mathbb{R}.$$

II. Lebsgue measure results

Khintchine : Let $\Psi : \mathbb{N} \to \mathbb{R}^+$ be a decreasing function. Consider :

$$\mathcal{L}_\Psi := \{\theta : \|n\theta\| < \Psi(n) \ i.o.\}$$

- $\mathcal{L}_\Psi$ is of Lebsgue measure zero if $\sum \Psi(n) < \infty$;
- $\mathcal{L}_\Psi$ is of Lebsgue measure full if $\sum \Psi(n) = \infty$.

Duffin-Schaefer 1941 Conjecture : If Ψ is not decreasing, then

$\mathcal{L}_\Psi$ is of Lebsgue measure full if $\sum \phi(n)\Psi(n)/n = \infty$,

where ϕ is the Euler function.

Haynes-Pollington-Velani (arXiv 2009) :

Yes, if $\sum \phi(n)(\Psi(n)/n)^{1+\epsilon} = \infty$.

III. Hausdorff Dimension

Define

$$\mathcal{L}_\beta(y) := \{\theta : ||n\theta - y|| < n^{-1/\beta} \quad i.o.\}$$

Jarník 1929, Besicovith 1934 : For $\beta \le 1$,

$$\dim_H(\mathcal{L}_\beta(0)) = 2\beta/(\beta+1).$$

Levesley 1998 : For any $y \in \mathbb{R}$, and $\beta \le 1$,

$$\dim_H(\mathcal{L}_\beta(y)) = 2\beta/(\beta+1).$$

Define

$$\mathcal{L}_\beta[\theta] := \{y : ||n\theta - y|| < n^{-1/\beta} \quad i.o.\}$$

Bernik-Dodson 1999 : For all $\theta \in \mathbb{R}$, and $\beta \le 1$,

$$\omega \cdot \beta \le \dim_H(\mathcal{L}_\beta[\theta]) \le \beta$$

where $\omega \le 1$ is a real number such that $||n\theta|| \ge n^{-1/\omega} \quad ev.$.

Bugeaud 2003, Troubetzkoy-Schmeling 2003 : For all $\theta \in \mathbb{R}$, $\beta \le 1$:

$$\dim_H(\mathcal{L}_\beta[\theta]) = \beta.$$

IV. Higher dimensional cases

Levesley 1998 :

- $\Psi : \mathbb{N} \to \mathbb{R}^+$ a decreasing function.
- $Y \in \mathbb{R}^\ell$, $n \in \mathbb{Z}^m$ non-zero, and $|n| := \max\{|n_i|\}$.

Define :

$$\mathcal{L}_\Psi(Y) := \left\{\Theta \in \mathbb{R}^{m\ell} : \| n\Theta - Y \| < \Psi(|n|) \;\; i.o. \;\; n \in \mathbb{Z}^m \setminus \{0\}\right\}.$$

We have

$$Leb(\mathcal{L}_\Psi(Y)) = \begin{cases} 0 & \text{if } \sum r^{m-1}\Psi(r)^\ell < \infty \\ total & \text{if } \sum r^{m-1}\Psi(r)^\ell = \infty \end{cases}.$$

Denote $\lambda := \liminf_{r\to\infty} \frac{-\log \Psi(r)}{r}$. For all $Y \in \mathbb{R}^\ell$,

$$\dim_H(\mathcal{L}_\Psi(Y)) = \begin{cases} (m-1)\ell + \dfrac{m+\ell}{\lambda+1} & \text{if } \lambda > \dfrac{m}{\ell} \\ m\ell & \text{if } \lambda \leq \dfrac{m}{\ell} \end{cases}$$

V. Uniformly approximated points

In general, we consider the sizes of the following sets :

$$\mathcal{U}_\beta(0) := \left\{\theta : \ \forall Q \gg 1, \ \exists \ 1 \le n \le Q, \ \|n\theta\| < Q^{-1/\beta}\right\},$$

$$\mathcal{U}_\beta(y) := \left\{\theta : \ \forall Q \gg 1, \ \exists \ 1 \le n \le Q, \ \|n\theta - y\| < Q^{-1/\beta}\right\}.$$

and for fixed θ,

$$\mathcal{U}_\beta[\theta] := \left\{y : \ \forall Q \gg 1, \ \exists \ 1 \le n \le Q, \ \|n\theta - y\| < Q^{-1/\beta}\right\}.$$

Remark :

$$\mathcal{U}_\beta(0) = \begin{cases} \mathbb{Q} & \text{if } \ \beta < 1 \\ \mathbb{R} & \text{if } \ \beta \ge 1. \end{cases}$$

VI. Relation with the hitting time

Let T_θ be the rotation on $\mathbb{R}/\mathbb{Z}$.

Define

$$\tau_r^\theta(x,y) = \inf\{n : T_\theta^n x \in B(y,r)\}.$$

Define the hitting time rates :

$$\underline{R}^\theta(x,y) := \liminf_{r\to 0} \frac{\log \tau_r^\theta(x,y)}{-\log r}, \qquad \overline{R}^\theta(x,y) := \limsup_{r\to 0} \frac{\log \tau_r^\theta(x,y)}{-\log r}.$$

We have

$$\mathcal{L}_\beta(y) \approx \{\theta : \underline{R}^\theta(0,y) \le \beta\}, \quad \mathcal{U}_\beta(\theta) \approx \{\theta : \overline{R}^\theta(0,y) \le \beta\},$$

and

$$\mathcal{L}_\beta[\theta] \approx \{y : \underline{R}^\theta(0,y) \le \beta\}, \quad \mathcal{U}_\beta[\theta] \approx \{y : \overline{R}^\theta(0,y) \le \beta\}.$$

VII. A known result in higher dimensional cases

Theorem (Y. Cheung (Ann. Math 2011))

For $\delta > 0$, the set

$$\left\{(\theta_1, \theta_2) : \forall Q \gg 1,\ 1 \leq \exists n \leq Q,\ \max\{\|n\theta_1\|, \|n\theta_2\|\} < \frac{\delta}{Q^{\frac{1}{2}}}\right\}. \quad (1)$$

is of Hausdorff dimension $4/3$.

Remark : The set is called **singular points set**. It has some important geometric meaning.

VIII. Our result

An irrational θ is of **type η** if $\eta = \sup\{\beta : \liminf_{j\to\infty} j^\beta \|j\theta\| = 0\}$.

Theorem (L, D.H. Kim)

Let θ be an irrational of type $\eta > 1$. Then $\dim_H(\mathcal{U}_\beta[\theta])$ equals to

$$\begin{cases} 1, & \beta \geq \eta, \\ \displaystyle\lim_{k\to\infty} \frac{\log(n_1^\beta \|n_1\theta\| \cdot n_2^\beta \|n_2\theta\| \cdots n_{k-1}^\beta \|n_{k-1}\theta\| \cdot n_k^{\beta+1})}{\log(n_k \|n_k\theta\|^{-1})}, & 1 \leq \beta < \eta, \\ -\displaystyle\lim_{k\to\infty} \frac{\log(n_1 \|n_1\theta\|^\beta \cdot n_2 \|n_2\theta\|^\beta \cdots n_{k-1} \|n_{k-1}\theta\|^\beta)}{\log(n_k \|n_k\theta\|^{-1})}, & \dfrac{1}{\eta} \leq \beta < 1, \\ 0, & \beta < \dfrac{1}{\eta}. \end{cases}$$

where n_k is the "maximal" increasing sequence of positive integers such that $n_k^\beta \|n_k\theta\| < 1$ if $1 \leq \beta < \eta$ and $n_k \|n_k\theta\|^\beta < 1$ if $1/\eta \leq \beta < 1$.

IX. Our result - Continued

Corollary (L, D.H. Kim)

For any irrational θ of type $\eta > 1$ we have

$$\frac{\eta\beta - 1}{\eta^2 - 1} \le \dim_H(\mathcal{U}_\beta[\theta]) \le \frac{\beta+1}{\eta+1}, \qquad 1 \le \beta < \eta,$$

$$0 \le \dim_H(\mathcal{U}_\beta[\theta]) \le \frac{\eta\beta - 1}{\eta^2 - 1}, \qquad \frac{1}{\eta} \le \beta < 1.$$

Remark : For the case $\beta < 1$, optimize the upper bound w.r.t. η :

$$\dim_H(\mathcal{U}_\beta[\theta]) \le \frac{\beta^2}{2(1 + \sqrt{1 - \beta^2})}.$$

Since $1 + \sqrt{1 - \beta^2} > 1$, we have $\dim_H(\mathcal{U}_\beta[\theta]) < \frac{\beta}{2}$.
Recall that for $\beta < 1$,

- $\mathcal{U}_\beta[\theta] \subset \mathcal{L}_\beta[\theta]$ except for a countable set of points.
- $\dim_H(\mathcal{L}_\beta[\theta]) = \beta$.

X. Our result - Examples, Special cases

Examples

(i) Let θ be of type $\eta > 1$ with $q_{k+1} > q_k^\eta$ for all k. Then

$$\dim_H(F) = (\eta\beta - 1)/(\eta^2 - 1), \quad 1/\eta < \beta < \eta.$$

(ii) Let θ be of type $\eta > 1$ with a sequence $\{k_i\}$ such that $q_{k_i+1} > q_{k_i}^\eta$ and that $a_{n+1} = 1$ for $n \neq k_i$, and $q_{k_{i+1}} > {q_{k_i}}^{2^i}$. Then

$$\dim_H(F) = (\beta + 1)/(\eta + 1), \quad 1 < \beta < \eta.$$

Special cases :

- If θ is of **type** ∞ (Liouville numbers), then

$$\dim_H(\mathcal{U}_\beta[\theta]) = 0 \quad \forall \beta.$$

- If θ is of **type** 1 :

1. if $\beta = 1$, the dimension is 1.
2. if $\beta < 1$, the dimension is 0 ? (work in progress).