Iteration of quasiregular tangent functions in three dimensions

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Joint work with Alastair Fletcher

Quasiregular mappings

Quasiregular functions on $\mathbb{R}^n$ generalize analytic functions on $\mathbb{C}$.

Definition

- A continuous function $f : U \to \mathbb{R}^n$ on a domain $U \subseteq \mathbb{R}^n$ is called quasiregular if $f \in W^1_{n,\mathrm{loc}}(U)$ and there exists $K \geq 1$ such that

$$\|Df(\mathbf{x})\|^n \leq K J_f(\mathbf{x}) \qquad \text{a.e. in } U.$$

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- More generally, a continuous function $f : \mathbb{R}^n \to \mathbb{R}^n \cup \{\infty\}$ is called quasiregular (or quasimeromorphic) if the set of poles $f^{-1}(\infty)$ is discrete and if f is quasiregular on $\mathbb{R}^n \setminus f^{-1}(\infty)$.

The Zorich mapping

The Zorich map $Z : \mathbb{R}^3 \to \mathbb{R}^3 \setminus \{\mathbf{0}\}$ is a quasiregular analogue of the exponential function. It can be defined as follows:

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$$h : [-\tfrac{\pi}{2}, \tfrac{\pi}{2}]^2 \to \{(x, y, z) : x^2 + y^2 + z^2 = 1,\ z \geq 0\}.$$

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3. Extend Z to all of $\mathbb{R}^3$ by repeatedly reflecting in planes.

The Zorich map is quasiregular on $\mathbb{R}^3$ and doubly-periodic with periods $(2\pi, 0, 0)$ and $(0, 2\pi, 0)$.

Trigonometric analogues

- Quasiregular maps of $\mathbb{R}^n$ which generalize the sine and cosine functions have been constructed by Drasin, by Mayer and by Bergweiler and Eremenko.
- By iterating their 'trigonometric' map, Bergweiler and Eremenko obtained a seemingly paradoxical decomposition of $\mathbb{R}^n$.
- We will construct and iterate a 3-dimensional quasiregular analogue of the meromorphic tangent function.

Construction of a generalized tangent mapping

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Define a sense-preserving Möbius map $A : \mathbb{R}^3 \to \mathbb{R}^3 \cup \{\infty\}$ by

$$A(x, y, z) = (0, 0, 1) + \frac{(2x, 2y, -2(z+1))}{x^2 + y^2 + (z+1)^2}.$$

We then define our 3-dimensional analogue of tangent by

$$T(\mathbf{x}) = (A \circ Z)(2\mathbf{x}).$$

Expressions for T

T contains embedded copies of the usual (complex) tangent function:

- $T(0, y, z) = (0,\ \mathrm{Re}(\tan(y + iz)),\ \mathrm{Im}(\tan(y + iz)))$,
- $T(x, 0, z) = (\mathrm{Re}(\tan(x + iz)),\ 0,\ \mathrm{Im}(\tan(x + iz)))$.

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If $M(x, y) = \max\{|x|, |y|\} \leq \pi/4$ and we write $\zeta = M(x, y) + iz$, then

$$T(x, y, z) = \left(\frac{x}{\sqrt{x^2 + y^2}} \mathrm{Re}(\tan \zeta), \frac{y}{\sqrt{x^2 + y^2}} \mathrm{Re}(\tan \zeta), \mathrm{Im}(\tan \zeta) \right).$$

Geometric properties of T

Comparing T with tan, the z-axis plays the role of the imaginary axis, while the xy-plane plays the role of the real axis.

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T is highly symmetric: If R is a reflection in a co-ordinate plane then

$$T(R(\mathbf{x})) = R(T(\mathbf{x})).$$

Iteration of tangent maps on $\mathbb{C}$

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Theorem (Devaney and Keen)

- *If* $0 < \lambda < 1$, *then* $J(\tau_\lambda) \subseteq \mathbb{R}$ *is locally a Cantor set. Attracting fixed point at origin.*
- *If* $\lambda = 1$, *then* $J(\tau_\lambda) = \mathbb{R}$. *Parabolic fixed point at origin.*
- *If* $\lambda > 1$, *then* $J(\tau_\lambda) = \mathbb{R}$. *Attracting fixed points at* $\pm i\xi_0$, *where* $\xi_0 > 0$ *solves* $\xi_0 = \lambda \tanh \xi_0$.

Dynamics of λT

For $\lambda > 0$ we put

$$T_\lambda(\mathbf{x}) = \lambda T(\mathbf{x}).$$

We iterate T_λ and aim to establish an analogue of the $\lambda \tan \zeta$ results.

First, we describe the behaviour on the upper and lower half-spaces.

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- *If $0 < \lambda < 1$, then T_λ has an attracting fixed point at the origin.*
- *If $0 < \lambda \leq 1$, then $T_\lambda^k(\mathbf{x}) \to \mathbf{0}$ on $\{(x, y, z) : z \neq 0\}$, as $k \to \infty$.*

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- *If* $0 < \lambda \leq 1$, *then* $T_\lambda^k(\mathbf{x}) \to \mathbf{0}$ *on* $\{(x, y, z) : z \neq 0\}$, *as* $k \to \infty$.
- *If* $\lambda > 1$, *then* T_λ *has attracting fixed points at* $(0, 0, \pm\xi_0)$, *where* $\xi_0 = \lambda \tanh \xi_0$, *and*

$$T_\lambda^k(\mathbf{x}) \to (0, 0, \pm\xi_0) \quad \text{on} \quad \{(x, y, z) : \pm z > 0\}.$$

What's a Julia set?

For a meromorphic function f with poles, the Julia set $J(f)$ satisfies

$$J(f) = \overline{O_f^-(\infty)} = \partial I(f),$$

where $I(f) = \{\zeta : f^k(\zeta) \to \infty \text{ as } k \to \infty\}$.

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For all $\lambda > 0$*,*

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If U is an open set that meets J then, for some $m > 0$,

$$T_\lambda^m(U) = (\mathbb{R}^3 \cup \infty) \setminus \{(0, 0, \pm\lambda)\}.$$

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J is contained in the closure of the set of periodic points of T_λ.

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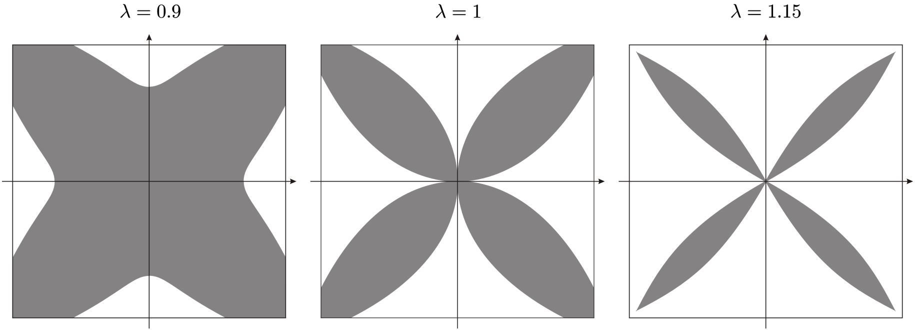
Theorem

If $\lambda > \sqrt{2}$ then $J = \{xy\text{-plane}\}$.
The constant $\sqrt{2}$ here cannot be replaced by any smaller value.

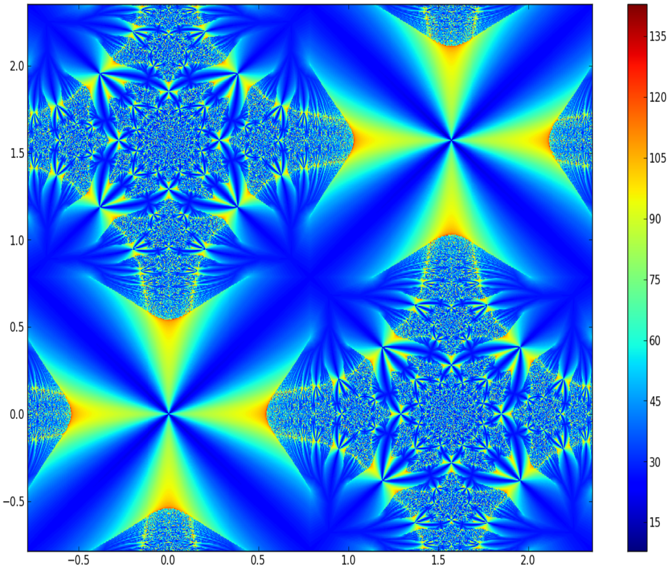
When $\lambda < \sqrt{2}$, a (relatively) open subset of the *xy*-plane lies in the attracting basin of $\mathbf{0}$ . . .

Attracting basin of **0**

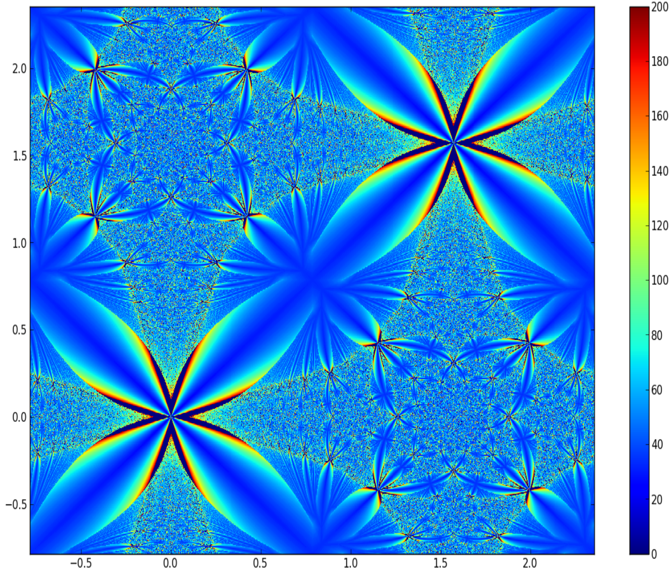


Each square is the subset $[-\frac{\pi}{4}, \frac{\pi}{4}]^2$ of the *xy*-plane.
The shaded points lie in the basin of attraction of **0**.

A numerical plot for $\lambda = 0.9$. Blue points $\to$ **0** fast, red points $\to$ **0** slow.



A numerical plot for $\lambda = 1$. Blue points $\to$ **0** fast, red points $\to$ **0** slow.



Around a pole for $\lambda = 0.7$. Thanks to Dan Goodman for code.