On a conjecture of Baker and a conjecture of Eremenko I

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Basic definitions

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The **Fatou set** (or stable set) is

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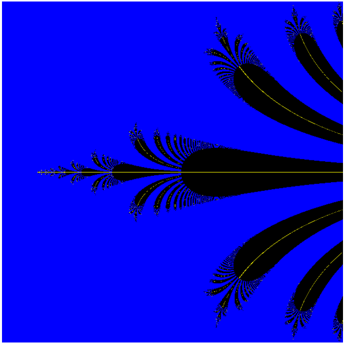
The **escaping set** is

$$I(f) = \{z : f^n(z) \to \infty \text{ as } n \to \infty\}.$$

Examples

Exponential functions



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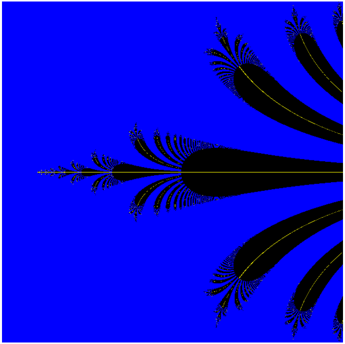

$f(z) = \lambda e^z$, $0 < \lambda < 1/e$

- $F(f)$ is an attracting basin
- $J(f)$ is a Cantor bouquet of curves
- $I(f) \subset J(f)$

A Fatou component U is a **wandering domain** if

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Theorem (Baker, 1984)

If U is a multiply connected Fatou component then

- *U is a wandering domain in $I(f)$*
- *$f^{n+1}(U)$ surrounds $f^n(U)$ for large n.*

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All components of $I(f)$ are unbounded.

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f is a finite composition of functions of finite order in class $\mathcal{B}$.

Theorem (Rippon and Stallard, 2005)

$I(f)$ is a "spider's web" if f has a multiply connected Fatou component.

Baker's Conjecture (1981)

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$$\rho = \limsup_{r\to\infty} \frac{\log\log M(r)}{\log r}.$$

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$$\rho = \limsup_{r\to\infty} \frac{\log\log M(r)}{\log r}.$$

If f has order less than 1 then

$$f(z) = cz^{p_0} \prod_{n\in\mathbb{N}} \left(1 + \frac{z}{a_n}\right)^{p_n},$$

where $p_n \in \{0, 1, 2, \ldots\}$ and $c, a_n \in \mathbb{C} \setminus \{0\}$.

Partial results on Baker's Conjecture

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Theorem (Rippon and Stallard, Hinkkanen and Miles (2009))

If f has very small growth then it has no unbounded Fatou components i.e. if

$$\log \log M(r) < \frac{\log r}{\log^m r}, \quad \text{for large } r, \text{ some } m \geq 2.$$

The link between the two conjectures

Definition

The **fast escaping set** is $A(f) = \bigcup_{L \in \mathbb{N}} f^{-L}(A_R(f))$ where:

$$A_R(f) = \{z \in \mathbb{C} : |f^n(z)| \geq M^n(R) \; \forall \; n \in \mathbb{N}\},$$

if $R > 0$ is such that $M^n(R) \to \infty$ as $n \to \infty$.

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Theorem (Rippon and Stallard, 2011)

If $A_R(f)$ is a spider's web then

- *$I(f)$ is a spider's web and so is connected;*

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All partial results on Baker's conjecture proved by methods which imply that $A_R(f)$ is a spider's web.

New progress on the two conjectures

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There exist infinite products of very small growth, with $\lambda > 0$ and $a_n > 0$, for which $A_R(f)$ is not a spider's web.

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Theorem

Let

$$f(z) = cz^{p_0} \prod_{n \in \mathbb{N}} \left(1 + \frac{z}{a_n}\right)^{p_n},$$

where $p_n \in \{0, 1, 2, \ldots\}$, $a_n > 0$ *and* $c \in \mathbb{R} \setminus \{0\}$, *be a function of order less than* $1/2$. *Then*

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- *f has no unbounded Fatou components;*
- *$I(f)$ is a spider's web.*

A sharp growth condition

Theorem

Let $R_n = M^n(R)$ *and* $\epsilon_n = \max_{R_n \le r \le R_{n+1}} \frac{\log\log M(r)}{\log r}$.

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Theorem

Suppose that $\log M(r) \le r^{\alpha}$, *where* $\alpha \in (0, 1/2)$.
Then there exists $t \in (r^{1-2\alpha}, r)$ *such that*

$$\log m(t) > \log M(r^{1-2\alpha}) - 2.$$

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and

$$\sum_{n=1}^{N} \epsilon_n < 3 \sum_{n=1}^{N} \delta_n, \text{ for large } N \in \mathbb{N}.$$