Selected results on measure-category products of ideals

Marek Balcerzak

Institute of Mathematics, Technical University of Łódź, ul. Wólczańska 215, 93-005 Łódź, Poland

Some notation

- $\mathcal{B}(X)$ – the family of Borel subsets of a Polish space X;

Some notation

- $\mathcal{B}(X)$ – the family of Borel subsets of a Polish space X;
- $\mathcal{M}$ – the σ-ideal of meager sets in $\mathbb{R}$;

Some notation

- $\mathcal{B}(X)$ – the family of Borel subsets of a Polish space X;
- $\mathcal{M}$ – the σ-ideal of meager sets in $\mathbb{R}$;
- $\mathcal{N}$ – the σ-ideal of Lebesgue null sets in $\mathbb{R}$;

Some notation

- $\mathcal{B}(X)$ – the family of Borel subsets of a Polish space X;
- $\mathcal{M}$ – the σ-ideal of meager sets in $\mathbb{R}$;
- $\mathcal{N}$ – the σ-ideal of Lebesgue null sets in $\mathbb{R}$;
- for $A \subseteq X^2$ and $x \in X$, let

Some notation

- $\mathcal{B}(X)$ – the family of Borel subsets of a Polish space X;
- $\mathcal{M}$ – the σ-ideal of meager sets in $\mathbb{R}$;
- $\mathcal{N}$ – the σ-ideal of Lebesgue null sets in $\mathbb{R}$;
- for $A \subseteq X^2$ and $x \in X$, let $A(x) = \{y \in X : (x, y) \in A\}$, this is x-section of A;

Some notation

- $\mathcal{B}(X)$ – the family of Borel subsets of a Polish space X;
- $\mathcal{M}$ – the σ-ideal of meager sets in $\mathbb{R}$;
- $\mathcal{N}$ – the σ-ideal of Lebesgue null sets in $\mathbb{R}$;
- for $A \subseteq X^2$ and $x \in X$, let $A(x) = \{y \in X \colon (x, y) \in A\}$, this is x-section of A;
- for two σ-ideals $\mathcal{I}, \mathcal{J} \subseteq \mathcal{P}(X)$,

Some notation

- $\mathcal{B}(X)$ – the family of Borel subsets of a Polish space X;
- $\mathcal{M}$ – the σ-ideal of meager sets in $\mathbb{R}$;
- $\mathcal{N}$ – the σ-ideal of Lebesgue null sets in $\mathbb{R}$;
- for $A \subseteq X^2$ and $x \in X$, let $A(x) = \{y \in X \colon (x, y) \in A\}$, this is x-section of A;
- for two σ-ideals $\mathcal{I}, \mathcal{J} \subseteq \mathcal{P}(X)$, define

$$\mathcal{I} \otimes \mathcal{J} := \{A \subseteq X^2 \colon (\exists B \in \mathcal{B}(X^2))(A \subseteq B \text{ and } \{x \colon B(x) \notin \mathcal{J}\} \in \mathcal{I})\},$$

Some notation

- $\mathcal{B}(X)$ – the family of Borel subsets of a Polish space X;
- $\mathcal{M}$ – the σ-ideal of meager sets in $\mathbb{R}$;
- $\mathcal{N}$ – the σ-ideal of Lebesgue null sets in $\mathbb{R}$;
- for $A \subseteq X^2$ and $x \in X$, let $A(x) = \{y \in X \colon (x, y) \in A\}$, this is x-section of A;
- for two σ-ideals $\mathcal{I}, \mathcal{J} \subseteq \mathcal{P}(X)$, define

  $$\mathcal{I} \otimes \mathcal{J} := \{A \subseteq X^2 \colon (\exists B \in \mathcal{B}(X^2))(A \subseteq B \text{ and } \{x \colon B(x) \notin \mathcal{J}\} \in \mathcal{I})\},$$

  this σ-ideal is called the **Fubini product** of $\mathcal{I}$ and $\mathcal{J}$.

In particular,

- $\mathcal{M} \otimes \mathcal{M}$ is the σ-ideal of meager sets in $\mathbb{R}^2$;

In particular,

- $\mathcal{M} \otimes \mathcal{M}$ is the σ-ideal of meager sets in $\mathbb{R}^2$;
- $\mathcal{N} \otimes \mathcal{N}$ is the σ-ideal of Lebesgue null sets in $\mathbb{R}^2$.

In particular,

- $\mathcal{M} \otimes \mathcal{M}$ is the σ-ideal of meager sets in $\mathbb{R}^2$;
- $\mathcal{N} \otimes \mathcal{N}$ is the σ-ideal of Lebesgue null sets in $\mathbb{R}^2$.

We will talk about the mixed product σ-ideals $\mathcal{M} \otimes \mathcal{N}$ and $\mathcal{N} \otimes \mathcal{M}$.

Some facts

Some facts

- $\mathcal{M}$ and $\mathcal{N}$ are *orthogonal*, i.e. there are disjoint sets $A \in \mathcal{M}$ and $B \in \mathcal{N}$ such that $A \cup B = \mathbb{R}$ [well known];

Some facts

- $\mathcal{M}$ and $\mathcal{N}$ are *orthogonal*, i.e. there are disjoint sets $A \in \mathcal{M}$ and $B \in \mathcal{N}$ such that $A \cup B = \mathbb{R}$ [well known];
- $\mathcal{M} \otimes \mathcal{N}$ and $\mathcal{N} \otimes \mathcal{M}$ are orthogonal [Mendez [1976]];

Some facts

- $\mathcal{M}$ and $\mathcal{N}$ are *orthogonal*, i.e. there are disjoint sets $A \in \mathcal{M}$ and $B \in \mathcal{N}$ such that $A \cup B = \mathbb{R}$ [well known];
- $\mathcal{M} \otimes \mathcal{N}$ and $\mathcal{N} \otimes \mathcal{M}$ are orthogonal [Mendez [1976]];
- $(\mathcal{M} \otimes \mathcal{M}) \cap (\mathcal{N} \otimes \mathcal{N})$ and $(\mathcal{M} \otimes \mathcal{N}) \cap (\mathcal{N} \otimes \mathcal{M})$ are orthogonal [Mendez [1976]];

Some facts

- $\mathcal{M}$ and $\mathcal{N}$ are *orthogonal*, i.e. there are disjoint sets $A \in \mathcal{M}$ and $B \in \mathcal{N}$ such that $A \cup B = \mathbb{R}$ [well known];
- $\mathcal{M} \otimes \mathcal{N}$ and $\mathcal{N} \otimes \mathcal{M}$ are orthogonal [Mendez [1976]];
- $(\mathcal{M} \otimes \mathcal{M}) \cap (\mathcal{N} \otimes \mathcal{N})$ and $(\mathcal{M} \otimes \mathcal{N}) \cap (\mathcal{N} \otimes \mathcal{M})$ are orthogonal [Mendez [1976]];
- $\mathcal{M} \otimes \mathcal{N}$ and $\mathcal{N} \otimes \mathcal{M}$ satisfy ccc, i.e. every disjoint family of Borel subsets of $\mathbb{R}^2$ that are not in the σ-ideal is countable [Fremlin (1987)];

Some facts

- $\mathcal{M}$ and $\mathcal{N}$ are *orthogonal*, i.e. there are disjoint sets $A \in \mathcal{M}$ and $B \in \mathcal{N}$ such that $A \cup B = \mathbb{R}$ [well known];
- $\mathcal{M} \otimes \mathcal{N}$ and $\mathcal{N} \otimes \mathcal{M}$ are orthogonal [Mendez [1976]];
- $(\mathcal{M} \otimes \mathcal{M}) \cap (\mathcal{N} \otimes \mathcal{N})$ and $(\mathcal{M} \otimes \mathcal{N}) \cap (\mathcal{N} \otimes \mathcal{M})$ are orthogonal [Mendez [1976]];
- $\mathcal{M} \otimes \mathcal{N}$ and $\mathcal{N} \otimes \mathcal{M}$ satisfy ccc, i.e. every disjoint family of Borel subsets of $\mathbb{R}^2$ that are not in the σ-ideal is countable [Fremlin (1987)];
- some asymmetry of cardinal coeficients:

Some facts

- $\mathcal{M}$ and $\mathcal{N}$ are *orthogonal*, i.e. there are disjoint sets $A \in \mathcal{M}$ and $B \in \mathcal{N}$ such that $A \cup B = \mathbb{R}$ [well known];
- $\mathcal{M} \otimes \mathcal{N}$ and $\mathcal{N} \otimes \mathcal{M}$ are orthogonal [Mendez [1976]];
- $(\mathcal{M} \otimes \mathcal{M}) \cap (\mathcal{N} \otimes \mathcal{N})$ and $(\mathcal{M} \otimes \mathcal{N}) \cap (\mathcal{N} \otimes \mathcal{M})$ are orthogonal [Mendez [1976]];
- $\mathcal{M} \otimes \mathcal{N}$ and $\mathcal{N} \otimes \mathcal{M}$ satisfy ccc, i.e. every disjoint family of Borel subsets of $\mathbb{R}^2$ that are not in the σ-ideal is countable [Fremlin (1987)];
- some asymmetry of cardinal coeficients:

$$\mathrm{cof}(\mathcal{N} \otimes \mathcal{M}) = \mathrm{cof}(\mathcal{N}), \quad \mathrm{add}(\mathcal{N} \otimes \mathcal{M}) = \mathrm{add}(\mathcal{N}),$$

Some facts

- $\mathcal{M}$ and $\mathcal{N}$ are *orthogonal*, i.e. there are disjoint sets $A \in \mathcal{M}$ and $B \in \mathcal{N}$ such that $A \cup B = \mathbb{R}$ [well known];
- $\mathcal{M} \otimes \mathcal{N}$ and $\mathcal{N} \otimes \mathcal{M}$ are orthogonal [Mendez [1976]];
- $(\mathcal{M} \otimes \mathcal{M}) \cap (\mathcal{N} \otimes \mathcal{N})$ and $(\mathcal{M} \otimes \mathcal{N}) \cap (\mathcal{N} \otimes \mathcal{M})$ are orthogonal [Mendez [1976]];
- $\mathcal{M} \otimes \mathcal{N}$ and $\mathcal{N} \otimes \mathcal{M}$ satisfy ccc, i.e. every disjoint family of Borel subsets of $\mathbb{R}^2$ that are not in the σ-ideal is countable [Fremlin (1987)];
- some asymmetry of cardinal coeficients:

$$\mathrm{cof}(\mathcal{N} \otimes \mathcal{M}) = \mathrm{cof}(\mathcal{N}), \quad \mathrm{add}(\mathcal{N} \otimes \mathcal{M}) = \mathrm{add}(\mathcal{N}),$$

$$\mathrm{cof}(\mathcal{M}\otimes\mathcal{N}) = \mathrm{cof}([\mathbb{R}]^{\leq\omega}) = \mathfrak{c}, \quad \mathrm{add}(\mathcal{M}\otimes\mathcal{N}) = \mathrm{add}([\mathbb{R}]^{\leq\omega}) = \omega_1$$

[Cichoń, Pawlikowski (1986), Fremlin (1991)];

- $\mathcal{M} \otimes \mathcal{N}$ and $\mathcal{N} \otimes \mathcal{M}$ are not invariant under nonzero rotations [M.B.+Sz. Głąb (2010)]; this answers a question of Natkaniec.

Structural properties

Again some notation:

Structural properties

Again some notation:

- $\triangle$ – the operation of symmetric difference;

Structural properties

Again some notation:

- $\triangle$ – the operation of symmetric difference;
- for $\mathcal{E}, \mathcal{I} \subseteq \mathcal{P}(X)$, let $\mathcal{E} * \mathcal{I} = \{B \triangle A \colon B \in \mathcal{E},\ A \in \mathcal{I}\}$;

Structural properties

Again some notation:

- $\triangle$ – the operation of symmetric difference;
- for $\mathcal{E}, \mathcal{I} \subseteq \mathcal{P}(X)$, let $\mathcal{E} * \mathcal{I} = \{B \triangle A \colon B \in \mathcal{E},\ A \in \mathcal{I}\}$;
- denote $\mathcal{F}_\sigma \sqcup \mathcal{G}_\delta = \{A \cup B \colon A \in \mathcal{F}_\sigma,\ B \in \mathcal{G}_\delta\ A, B \subseteq X\}$.

Theorem (Fremlin (1991), M.B. (1992))

Theorem (Fremlin (1991), M.B. (1992))

Let $\mathcal{I} \in \{\mathcal{N} \otimes \mathcal{M}, \mathcal{M} \otimes \mathcal{N}\}$.

Theorem (Fremlin (1991), M.B. (1992))

Let $\mathcal{I} \in \{\mathcal{N} \otimes \mathcal{M}, \mathcal{M} \otimes \mathcal{N}\}$*. Then each set from* $\mathcal{I}$ *is contained in a set from* $\mathcal{I}$ *of class* $\mathcal{F}_\sigma \sqcup \mathcal{G}_\delta$*. Additionally,*

Theorem (Fremlin (1991), M.B. (1992))

Let $\mathcal{I} \in \{\mathcal{N} \otimes \mathcal{M}, \mathcal{M} \otimes \mathcal{N}\}$. *Then each set from* $\mathcal{I}$ *is contained in a set from* $\mathcal{I}$ *of class* $\mathcal{F}_\sigma \sqcup \mathcal{G}_\delta$. *Additionally,*

- *for* $\mathcal{I} = \mathcal{N} \otimes \mathcal{M}$ *we have* $\mathcal{B} * \mathcal{I} = \mathcal{F}_\sigma * \mathcal{I}$;

Theorem (Fremlin (1991), M.B. (1992))

Let $\mathcal{I} \in \{\mathcal{N} \otimes \mathcal{M}, \mathcal{M} \otimes \mathcal{N}\}$. *Then each set from* $\mathcal{I}$ *is contained in a set from* $\mathcal{I}$ *of class* $\mathcal{F}_\sigma \sqcup \mathcal{G}_\delta$. *Additionally,*

- *for* $\mathcal{I} = \mathcal{N} \otimes \mathcal{M}$ *we have* $\mathcal{B} * \mathcal{I} = \mathcal{F}_\sigma * \mathcal{I}$;
- *for* $\mathcal{I} = \mathcal{M} \otimes \mathcal{N}$ *we have* $\mathcal{B} * \mathcal{I} = (\mathcal{F}_\sigma \sqcup \mathcal{G}_\delta) * \mathcal{I}$.

Theorem (M.B.+Sz. Głąb (2010))

Let $f : \mathbb{R}^2 \to \mathbb{R}$ *be a Borel function. Then*

Application 1: Borel functions of two variables

Theorem (M.B.+Sz. Głąb (2010))

Let $f : \mathbb{R}^2 \to \mathbb{R}$ *be a Borel function. Then*

(a) *there exists a set* $D \in \mathcal{N} \otimes \mathcal{M}$ *such that* $h = f|(\mathbb{R}^2 \setminus D)$ *is Borel measurable of class 1*

Application 1: Borel functions of two variables

Theorem (M.B.+Sz. Głąb (2010))

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a Borel function. Then

(a) *there exists a set $D \in \mathcal{N} \otimes \mathcal{M}$ such that $h = f|(\mathbb{R}^2 \setminus D)$ is Borel measurable of class 1 with $\mathcal{N}$-almost all sections h_x continuous;*

Theorem (M.B.+Sz. Głąb (2010))

Let $f : \mathbb{R}^2 \to \mathbb{R}$ *be a Borel function. Then*

(a) *there exists a set* $D \in \mathcal{N} \otimes \mathcal{M}$ *such that* $h = f|(\mathbb{R}^2 \setminus D)$ *is Borel measurable of class 1 with* $\mathcal{N}$*-almost all sections* h_x *continuous;*

(b) *there exists a set* $D \in \mathcal{M} \otimes \mathcal{N}$ *such that* $h = f|(\mathbb{R}^2 \setminus D)$ *is Borel measurable of class 1.*

Application 2: monotone hull operations

Consider a triple $(X, \mathcal{S}, \mathcal{I})$ where $\mathcal{S} \subseteq \mathcal{P}(X)$ is a σ-algebra and $\mathcal{I} \subseteq \mathcal{S}$ is a σ-ideal.

Application 2: monotone hull operations

Consider a triple $(X, \mathcal{S}, \mathcal{I})$ where $\mathcal{S} \subseteq \mathcal{P}(X)$ is a σ-algebra and $\mathcal{I} \subseteq \mathcal{S}$ is a σ-ideal. Fix two families $\mathcal{A} \subseteq \mathcal{P}(X)$ and $\mathcal{H} \subseteq \mathcal{S}$.

Application 2: monotone hull operations

Consider a triple $(X, \mathcal{S}, \mathcal{I})$ where $\mathcal{S} \subseteq \mathcal{P}(X)$ is a σ-algebra and $\mathcal{I} \subseteq \mathcal{S}$ is a σ-ideal. Fix two families $\mathcal{A} \subseteq \mathcal{P}(X)$ and $\mathcal{H} \subseteq \mathcal{S}$. We say that:

Application 2: monotone hull operations

Consider a triple $(X, \mathcal{S}, \mathcal{I})$ where $\mathcal{S} \subseteq \mathcal{P}(X)$ is a σ-algebra and $\mathcal{I} \subseteq \mathcal{S}$ is a σ-ideal. Fix two families $\mathcal{A} \subseteq \mathcal{P}(X)$ and $\mathcal{H} \subseteq \mathcal{S}$. We say that:

- $H \in \mathcal{H}$ is an $\mathcal{H}$-*hull* of $A \subseteq X$ if $A \subseteq H$ and for every $G \in \mathcal{H}$ with $A \subseteq G$ we have $H \setminus G \in \mathcal{I}$;

Application 2: monotone hull operations

Consider a triple $(X, \mathcal{S}, \mathcal{I})$ where $\mathcal{S} \subseteq \mathcal{P}(X)$ is a σ-algebra and $\mathcal{I} \subseteq \mathcal{S}$ is a σ-ideal. Fix two families $\mathcal{A} \subseteq \mathcal{P}(X)$ and $\mathcal{H} \subseteq \mathcal{S}$. We say that:

- $H \in \mathcal{H}$ is an $\mathcal{H}$-*hull* of $A \subseteq X$ if $A \subseteq H$ and for every $G \in \mathcal{H}$ with $A \subseteq G$ we have $H \setminus G \in \mathcal{I}$;
- $\varphi\colon \mathcal{A} \to \mathcal{H}$ is an $\mathcal{H}$-*hull operation* on $\mathcal{A}$ if for each $A \in \mathcal{A}$, $\varphi(A)$ is an $\mathcal{H}$-hull of A;

Application 2: monotone hull operations

Consider a triple $(X, \mathcal{S}, \mathcal{I})$ where $\mathcal{S} \subseteq \mathcal{P}(X)$ is a σ-algebra and $\mathcal{I} \subseteq \mathcal{S}$ is a σ-ideal. Fix two families $\mathcal{A} \subseteq \mathcal{P}(X)$ and $\mathcal{H} \subseteq \mathcal{S}$. We say that:

- $H \in \mathcal{H}$ is an $\mathcal{H}$-*hull* of $A \subseteq X$ if $A \subseteq H$ and for every $G \in \mathcal{H}$ with $A \subseteq G$ we have $H \setminus G \in \mathcal{I}$;
- $\varphi\colon \mathcal{A} \to \mathcal{H}$ is an $\mathcal{H}$-*hull operation* on $\mathcal{A}$ if for each $A \in \mathcal{A}$, $\varphi(A)$ is an $\mathcal{H}$-hull of A;
- an $\mathcal{H}$-hull operation on $\mathcal{A}$ is *monotone* if for all $A, B \in \mathcal{A}$ with $A \subseteq B$ we have $\varphi(A) \subseteq \varphi(B)$.

Application 2: monotone hull operations

Consider a triple $(X, \mathcal{S}, \mathcal{I})$ where $\mathcal{S} \subseteq \mathcal{P}(X)$ is a σ-algebra and $\mathcal{I} \subseteq \mathcal{S}$ is a σ-ideal. Fix two families $\mathcal{A} \subseteq \mathcal{P}(X)$ and $\mathcal{H} \subseteq \mathcal{S}$. We say that:

- $H \in \mathcal{H}$ is an $\mathcal{H}$-*hull* of $A \subseteq X$ if $A \subseteq H$ and for every $G \in \mathcal{H}$ with $A \subseteq G$ we have $H \setminus G \in \mathcal{I}$;
- $\varphi \colon \mathcal{A} \to \mathcal{H}$ is an $\mathcal{H}$-*hull operation* on $\mathcal{A}$ if for each $A \in \mathcal{A}$, $\varphi(A)$ is an $\mathcal{H}$-hull of A;
- an $\mathcal{H}$-hull operation on $\mathcal{A}$ is *monotone* if for all $A, B \in \mathcal{A}$ with $A \subseteq B$ we have $\varphi(A) \subseteq \varphi(B)$.

Elekes and Máthé (2009) answered the following question:
Does there exist a monotone Borel-hull operation on the σ-algebra of Lebesgue measurable sets?

Application 2: monotone hull operations

Consider a triple $(X, \mathcal{S}, \mathcal{I})$ where $\mathcal{S} \subseteq \mathcal{P}(X)$ is a σ-algebra and $\mathcal{I} \subseteq \mathcal{S}$ is a σ-ideal. Fix two families $\mathcal{A} \subseteq \mathcal{P}(X)$ and $\mathcal{H} \subseteq \mathcal{S}$. We say that:

- $H \in \mathcal{H}$ is an $\mathcal{H}$-*hull* of $A \subseteq X$ if $A \subseteq H$ and for every $G \in \mathcal{H}$ with $A \subseteq G$ we have $H \setminus G \in \mathcal{I}$;
- $\varphi \colon \mathcal{A} \to \mathcal{H}$ is an $\mathcal{H}$-*hull operation* on $\mathcal{A}$ if for each $A \in \mathcal{A}$, $\varphi(A)$ is an $\mathcal{H}$-hull of A;
- an $\mathcal{H}$-hull operation on $\mathcal{A}$ is *monotone* if for all $A, B \in \mathcal{A}$ with $A \subseteq B$ we have $\varphi(A) \subseteq \varphi(B)$.

Elekes and Máthé (2009) answered the following question:
Does there exist a monotone Borel-hull operation on the σ-algebra of Lebesgue measurable sets?
One can pose similar questions for the Baire category case and for σ-algebras associated with $\mathcal{M} \otimes \mathcal{N}$ and $\mathcal{N} \otimes \mathcal{M}$.

We have two series of independence-type results.

Theorem

(a) *In a model obtained by adding ω_2 Cohen reals to a model satifying CH there is no monotone Borel hull on $\mathcal{N}$.* [Elekes and Máthé (2009)]

We have two series of independence-type results.

Theorem

(a) *In a model obtained by adding ω_2 Cohen reals to a model satifying CH there is no monotone Borel hull on $\mathcal{N}$. [Elekes and Máthé (2009)]*

(b) *In a model obtained by adding ω_2 random reals to a model satifying CH there is no monotone Borel hull on $\mathcal{N}$. [M.B.+T. Filipczak (2011)]*

We have two series of independence-type results.

Theorem

(a) *In a model obtained by adding ω_2 Cohen reals to a model satifying CH there is no monotone Borel hull on $\mathcal{N}$. [Elekes and Máthé (2009)]*

(b) *In a model obtained by adding ω_2 random reals to a model satifying CH there is no monotone Borel hull on $\mathcal{N}$. [M.B.+T. Filipczak (2011)]*

(c) *Let $\mathcal{I} \in \{\mathcal{M} \otimes \mathcal{N}, \mathcal{N} \otimes \mathcal{M}\}$. Consider a model obtained by adding either ω_2 Cohen reals or ω_2 random reals to a model satisfying CH. Then there is no monotone Borel hull on $\mathcal{I}$. [M.B.+T. Filipczak (2011)]*

We have two series of independence-type results.

Theorem

(a) *In a model obtained by adding ω_2 Cohen reals to a model satifying CH there is no monotone Borel hull on $\mathcal{N}$.* *[Elekes and Máthé (2009)]*

(b) *In a model obtained by adding ω_2 random reals to a model satifying CH there is no monotone Borel hull on $\mathcal{N}$.* *[M.B.+T. Filipczak (2011)]*

(c) *Let $\mathcal{I} \in \{\mathcal{M} \otimes \mathcal{N}, \mathcal{N} \otimes \mathcal{M}\}$. Consider a model obtained by adding either ω_2 Cohen reals or ω_2 random reals to a model satisfying CH. Then there is no monotone Borel hull on $\mathcal{I}$.* *[M.B.+T. Filipczak (2011)]*

Proof.

Statement (c) follows from (a), (b) and the equalities

$$\text{non}(\mathcal{M} \otimes \mathcal{N}) = \max\{\text{non}(\mathcal{M}), \text{non}(\mathcal{N})\} = \text{non}(\mathcal{N} \otimes \mathcal{M}).$$

Theorem (CH)

(a) *There exists a monotone* $\mathcal{F}_{\sigma\delta\sigma}$*-hull operation on* $\mathcal{B}(\mathbb{R}) * \mathcal{N}$.
[Elekes and Máthé (2009)]

Theorem (**CH**)

(a) *There exists a monotone $\mathcal{F}_{\sigma\delta\sigma}$-hull operation on $\mathcal{B}(\mathbb{R}) * \mathcal{N}$.* *[Elekes and Máthé (2009)]*

(b) *There exists a monotone $\mathcal{G}_{\delta\sigma}$-hull operation on $\mathcal{B}(\mathbb{R}) * \mathcal{M}$.* *[M.B.+T. Filipczak (2011)]*

Theorem (**CH**)

(a) *There exists a monotone $\mathcal{F}_{\sigma\delta\sigma}$-hull operation on $\mathcal{B}(\mathbb{R}) * \mathcal{N}$.* *[Elekes and Máthé (2009)]*

(b) *There exists a monotone $\mathcal{G}_{\delta\sigma}$-hull operation on $\mathcal{B}(\mathbb{R}) * \mathcal{M}$.* *[M.B.+T. Filipczak (2011)]*

(c) *Let $\mathcal{I} \in \{\mathcal{M} \otimes \mathcal{N}, \mathcal{N} \otimes \mathcal{M}\}$. There exists a monotone $\mathcal{G}_{\delta\sigma\delta\sigma}$-hull operation on $\mathcal{B}(\mathbb{R}^2) * \mathcal{I}$.* *[M.B.+T. Filipczak (2011)]*

Theorem (**CH**)

(a) *There exists a monotone $\mathcal{F}_{\sigma\delta\sigma}$-hull operation on $\mathcal{B}(\mathbb{R}) * \mathcal{N}$.* *[Elekes and Máthé (2009)]*

(b) *There exists a monotone $\mathcal{G}_{\delta\sigma}$-hull operation on $\mathcal{B}(\mathbb{R}) * \mathcal{M}$.* *[M.B.+T. Filipczak (2011)]*

(c) *Let $\mathcal{I} \in \{\mathcal{M} \otimes \mathcal{N}, \mathcal{N} \otimes \mathcal{M}\}$. There exists a monotone $\mathcal{G}_{\delta\sigma\delta\sigma}$-hull operation on $\mathcal{B}(\mathbb{R}^2) * \mathcal{I}$.* *[M.B.+T. Filipczak (2011)]*

Moreover, in all these cases, the respective monotone hull operations exist on the whole power set $\mathcal{P}(\mathbb{R})$ or $\mathcal{P}(\mathbb{R}^2)$.

Application 3: density-type topologies

Definition

We say that $x \in \mathbb{R}$ is a *density point* of a measurable set $A \subseteq \mathbb{R}$ if

$$\lim_{h \to 0^+} \frac{\lambda(A \cap [x-h, x+h])}{2h} = 1$$

where λ is Lebesgue measure of linear sets.

Application 3: density-type topologies

Definition

We say that $x \in \mathbb{R}$ is a *density point* of a measurable set $A \subseteq \mathbb{R}$ if

$$\lim_{h \to 0^+} \frac{\lambda(A \cap [x-h, x+h])}{2h} = 1$$

where λ is Lebesgue measure of linear sets.

Proposition

[Wilczyński (1984/85)] For a measurable set $A \subseteq \mathbb{R}$, the following conditions are equivalent:

Application 3: density-type topologies

Definition

We say that $x \in \mathbb{R}$ is a *density point* of a measurable set $A \subseteq \mathbb{R}$ if

$$\lim_{h \to 0^+} \frac{\lambda(A \cap [x-h, x+h])}{2h} = 1$$

where λ is Lebesgue measure of linear sets.

Proposition

[Wilczyński (1984/85)] For a measurable set $A \subseteq \mathbb{R}$, the following conditions are equivalent:

- *x is a density point of A;*

Application 3: density-type topologies

Definition

We say that $x \in \mathbb{R}$ is a *density point* of a measurable set $A \subseteq \mathbb{R}$ if

$$\lim_{h \to 0^+} \frac{\lambda(A \cap [x-h, x+h])}{2h} = 1$$

where λ is Lebesgue measure of linear sets.

Proposition

[Wilczyński (1984/85)] For a measurable set $A \subseteq \mathbb{R}$, the following conditions are equivalent:

- *x is a density point of A;*
- *each increasing sequence $(n_m)_{m \in \mathbb{N}}$ has subsequence $(n_{m_k})_{k \in \mathbb{N}}$ such that*

$$[-1, 1] \setminus \liminf_{k \to \infty} (n_{m_k}(A - x)) \in \mathcal{N}.$$

Here $nA := \{na\colon a \in A\}$ and $A - x := \{a - x\colon a \in A\}$.
This definition was used by Wilczyński (1984/85) to create the category analogue of density topology.

Here $nA := \{na\colon a \in A\}$ and $A - x := \{a - x\colon a \in A\}$.
This definition was used by Wilczyński (1984/85) to create the category analogue of density topology. A similar scheme leads to density-type topologies generated by $\mathcal{M} \otimes \mathcal{N}$ and $\mathcal{N} \otimes \mathcal{M}$.

Here $nA := \{na \colon a \in A\}$ and $A - x := \{a - x \colon a \in A\}$.
This definition was used by Wilczyński (1984/85) to create the category analogue of density topology. A similar scheme leads to density-type topologies generated by $\mathcal{M} \otimes \mathcal{N}$ and $\mathcal{N} \otimes \mathcal{M}$.

Definition

[M.B+J. Hejduk (1994/95)] $(0,0)$ is called an $\mathcal{M} \otimes \mathcal{N}$-*density point* of $A \in \mathcal{B} * (\mathcal{M} \otimes \mathcal{N})$ if

Here $nA := \{na\colon a \in A\}$ and $A - x := \{a - x\colon a \in A\}$.
This definition was used by Wilczyński (1984/85) to create the category analogue of density topology. A similar scheme leads to density-type topologies generated by $\mathcal{M} \otimes \mathcal{N}$ and $\mathcal{N} \otimes \mathcal{M}$.

Definition

[M.B+J. Hejduk (1994/95)] $(0,0)$ is called an $\mathcal{M} \otimes \mathcal{N}$-*density point* of $A \in \mathcal{B} * (\mathcal{M} \otimes \mathcal{N})$ if for each increasing sequence $(n_m)_{m\in\mathbb{N}}$ there are a subsequence $(n_{m_k})_{k\in\mathbb{N}}$ and a set $E \in \mathcal{M}$ such that

$$[-1,1] \setminus \liminf_{k\to\infty}(n_{m_k}A(t/n_{m_k})) \in \mathcal{N} \text{ for each } t \in [-1,1] \setminus E.$$

Here $nA := \{na\colon a \in A\}$ and $A - x := \{a - x\colon a \in A\}$.
This definition was used by Wilczyński (1984/85) to create the category analogue of density topology. A similar scheme leads to density-type topologies generated by $\mathcal{M} \otimes \mathcal{N}$ and $\mathcal{N} \otimes \mathcal{M}$.

Definition

[M.B+J. Hejduk (1994/95)] $(0,0)$ is called an $\mathcal{M} \otimes \mathcal{N}$-*density point* of $A \in \mathcal{B} * (\mathcal{M} \otimes \mathcal{N})$ if for each increasing sequence $(n_m)_{m\in\mathbb{N}}$ there are a subsequence $(n_{m_k})_{k\in\mathbb{N}}$ and a set $E \in \mathcal{M}$ such that

$$[-1,1] \setminus \liminf_{k\to\infty}(n_{m_k}A(t/n_{m_k})) \in \mathcal{N} \text{ for each } t \in [-1,1] \setminus E.$$

- In the standard way, we extend this definition from the case of $(0,0)$ to the case of an arbitrary $(x,y) \in \mathbb{R}^2$.

Here $nA := \{na\colon a \in A\}$ and $A - x := \{a - x\colon a \in A\}$.
This definition was used by Wilczyński (1984/85) to create the category analogue of density topology. A similar scheme leads to density-type topologies generated by $\mathcal{M} \otimes \mathcal{N}$ and $\mathcal{N} \otimes \mathcal{M}$.

Definition

[M.B+J. Hejduk (1994/95)] $(0,0)$ is called an $\mathcal{M} \otimes \mathcal{N}$-*density point* of $A \in \mathcal{B} * (\mathcal{M} \otimes \mathcal{N})$ if for each increasing sequence $(n_m)_{m\in\mathbb{N}}$ there are a subsequence $(n_{m_k})_{k\in\mathbb{N}}$ and a set $E \in \mathcal{M}$ such that

$$[-1,1] \setminus \liminf_{k\to\infty}(n_{m_k}A(t/n_{m_k})) \in \mathcal{N} \text{ for each } t \in [-1,1] \setminus E.$$

- In the standard way, we extend this definition from the case of $(0,0)$ to the case of an arbitrary $(x,y) \in \mathbb{R}^2$.
- The definition of an $\mathcal{N} \otimes \mathcal{M}$-density point is analogous.

Method of construction of a density-type topology from a notion of density point

- Fix a σ-algebra $\mathcal{S} \subseteq \mathcal{P}(\mathbb{R}^k)$ and a σ-ideal $\mathcal{J} \subseteq \mathcal{S}$ that are invariant under translations and dilations.

Method of construction of a density-type topology from a notion of density point

- Fix a σ-algebra $\mathcal{S} \subseteq \mathcal{P}(\mathbb{R}^k)$ and a σ-ideal $\mathcal{J} \subseteq \mathcal{S}$ that are invariant under translations and dilations.
- Assume that we have a notion of an $\mathcal{J}$-density point. For $A \in \mathcal{S}$ let

Method of construction of a density-type topology from a notion of density point

- Fix a σ-algebra $\mathcal{S} \subseteq \mathcal{P}(\mathbb{R}^k)$ and a σ-ideal $\mathcal{J} \subseteq \mathcal{S}$ that are invariant under translations and dilations.
- Assume that we have a notion of an $\mathcal{J}$-density point. For $A \in \mathcal{S}$ let

$$\Phi(A) = \text{the set of } \mathcal{J}\text{-density points of } A.$$

Method of construction of a density-type topology from a notion of density point

- Fix a σ-algebra $\mathcal{S} \subseteq \mathcal{P}(\mathbb{R}^k)$ and a σ-ideal $\mathcal{J} \subseteq \mathcal{S}$ that are invariant under translations and dilations.
- Assume that we have a notion of an $\mathcal{J}$-density point. For $A \in \mathcal{S}$ let

$$\Phi(A) = \text{the set of } \mathcal{J}\text{-density points of } A.$$

- Assume that $\Phi \colon \mathcal{S} \to \mathcal{P}(\mathbb{R}^k)$ is a lower density operator, that is

Method of construction of a density-type topology from a notion of density point

- Fix a σ-algebra $\mathcal{S} \subseteq \mathcal{P}(\mathbb{R}^k)$ and a σ-ideal $\mathcal{J} \subseteq \mathcal{S}$ that are invariant under translations and dilations.
- Assume that we have a notion of an $\mathcal{J}$-density point. For $A \in \mathcal{S}$ let

$$\Phi(A) = \text{the set of } \mathcal{J}\text{-density points of } A.$$

- Assume that $\Phi\colon \mathcal{S} \to \mathcal{P}(\mathbb{R}^k)$ is a lower density operator, that is
  (1) $\Phi(A) \in \mathcal{S}$ for each $A \in \mathcal{S}$;

Method of construction of a density-type topology from a notion of density point

- Fix a σ-algebra $\mathcal{S} \subseteq \mathcal{P}(\mathbb{R}^k)$ and a σ-ideal $\mathcal{J} \subseteq \mathcal{S}$ that are invariant under translations and dilations.
- Assume that we have a notion of an $\mathcal{J}$-density point. For $A \in \mathcal{S}$ let

$$\Phi(A) = \text{the set of } \mathcal{J}\text{-density points of } A.$$

- Assume that $\Phi \colon \mathcal{S} \to \mathcal{P}(\mathbb{R}^k)$ is a lower density operator, that is
  (1) $\Phi(A) \in \mathcal{S}$ for each $A \in \mathcal{S}$;
  (2) $\Phi(\emptyset) = \emptyset, \quad \Phi(\mathbb{R}^k) = \mathbb{R}^k$;

Method of construction of a density-type topology from a notion of density point

- Fix a σ-algebra $\mathcal{S} \subseteq \mathcal{P}(\mathbb{R}^k)$ and a σ-ideal $\mathcal{I} \subseteq \mathcal{S}$ that are invariant under translations and dilations.
- Assume that we have a notion of an $\mathcal{I}$-density point. For $A \in \mathcal{S}$ let

$$\Phi(A) = \text{the set of } \mathcal{I}\text{-density points of } A.$$

- Assume that $\Phi\colon \mathcal{S} \to \mathcal{P}(\mathbb{R}^k)$ is a lower density operator, that is
  (1) $\Phi(A) \in \mathcal{S}$ for each $A \in \mathcal{S}$;
  (2) $\Phi(\emptyset) = \emptyset, \quad \Phi(\mathbb{R}^k) = \mathbb{R}^k$;
  (3) $\Phi(A \cap B) = \Phi(A) \cap \Phi(B)$ for all $A, B \in \mathcal{S}$;

Method of construction of a density-type topology from a notion of density point

- Fix a σ-algebra $\mathcal{S} \subseteq \mathcal{P}(\mathbb{R}^k)$ and a σ-ideal $\mathcal{J} \subseteq \mathcal{S}$ that are invariant under translations and dilations.
- Assume that we have a notion of an $\mathcal{J}$-density point. For $A \in \mathcal{S}$ let

$$\Phi(A) = \text{the set of } \mathcal{J}\text{-density points of } A.$$

- Assume that $\Phi \colon \mathcal{S} \to \mathcal{P}(\mathbb{R}^k)$ is a lower density operator, that is
  (1) $\Phi(A) \in \mathcal{S}$ for each $A \in \mathcal{S}$;
  (2) $\Phi(\emptyset) = \emptyset, \quad \Phi(\mathbb{R}^k) = \mathbb{R}^k$;
  (3) $\Phi(A \cap B) = \Phi(A) \cap \Phi(B)$ for all $A, B \in \mathcal{S}$;
  (4) for all $A, B \in \mathcal{S}$, if $A \triangle B \in \mathcal{J}$ then $\Phi(A) = \Phi(B)$;

Method of construction of a density-type topology from a notion of density point

- Fix a σ-algebra $\mathcal{S} \subseteq \mathcal{P}(\mathbb{R}^k)$ and a σ-ideal $\mathcal{J} \subseteq \mathcal{S}$ that are invariant under translations and dilations.
- Assume that we have a notion of an $\mathcal{J}$-density point. For $A \in \mathcal{S}$ let

$$\Phi(A) = \text{the set of } \mathcal{J}\text{-density points of } A.$$

- Assume that $\Phi\colon \mathcal{S} \to \mathcal{P}(\mathbb{R}^k)$ is a lower density operator, that is
  (1) $\Phi(A) \in \mathcal{S}$ for each $A \in \mathcal{S}$;
  (2) $\Phi(\emptyset) = \emptyset, \quad \Phi(\mathbb{R}^k) = \mathbb{R}^k$;
  (3) $\Phi(A \cap B) = \Phi(A) \cap \Phi(B)$ for all $A, B \in \mathcal{S}$;
  (4) for all $A, B \in \mathcal{S}$, if $A \triangle B \in \mathcal{J}$ then $\Phi(A) = \Phi(B)$;
  (5) $\Phi(A) \triangle A \in \mathcal{J}$ for each $A \in \mathcal{S}$ (Lebesgue condition).

Method of construction of a density-type topology from a notion of density point

- Fix a σ-algebra $\mathcal{S} \subseteq \mathcal{P}(\mathbb{R}^k)$ and a σ-ideal $\mathcal{J} \subseteq \mathcal{S}$ that are invariant under translations and dilations.
- Assume that we have a notion of an $\mathcal{J}$-density point. For $A \in \mathcal{S}$ let

$$\Phi(A) = \text{the set of } \mathcal{J}\text{-density points of } A.$$

- Assume that $\Phi\colon \mathcal{S} \to \mathcal{P}(\mathbb{R}^k)$ is a lower density operator, that is
  (1) $\Phi(A) \in \mathcal{S}$ for each $A \in \mathcal{S}$;
  (2) $\Phi(\emptyset) = \emptyset, \quad \Phi(\mathbb{R}^k) = \mathbb{R}^k$;
  (3) $\Phi(A \cap B) = \Phi(A) \cap \Phi(B)$ for all $A, B \in \mathcal{S}$;
  (4) for all $A, B \in \mathcal{S}$, if $A \triangle B \in \mathcal{J}$ then $\Phi(A) = \Phi(B)$;
  (5) $\Phi(A) \triangle A \in \mathcal{J}$ for each $A \in \mathcal{S}$ (Lebesgue condition).
- Then $\tau_{\mathcal{J}} := \{A \in \mathcal{S} \colon A \subseteq \Phi(A)\}$ forms a topology.

For $\mathcal{J} \in \{\mathcal{M} \otimes \mathcal{N}, \mathcal{N} \otimes \mathcal{M}\}$ and $\mathcal{S} := \mathcal{B}(\mathbb{R}^2) * \mathcal{J}$, all conditions except for (5) can be checked by usual methods.

For $\mathcal{I} \in \{\mathcal{M} \otimes \mathcal{N}, \mathcal{N} \otimes \mathcal{M}\}$ and $\mathcal{S} := \mathcal{B}(\mathbb{R}^2) * \mathcal{I}$, all conditions except for (5) can be checked by usual methods.
The fulfillment of (5) is a nontrivial result [M.B+J. Hejduk (1994/95)].

For $\mathcal{I} \in \{\mathcal{M} \otimes \mathcal{N}, \mathcal{N} \otimes \mathcal{M}\}$ and $\mathcal{S} := \mathcal{B}(\mathbb{R}^2) * \mathcal{I}$, all conditions except for (5) can be checked by usual methods.
The fulfillment of (5) is a nontrivial result [M.B+J. Hejduk (1994/95)].
This leads to topologies $\tau_{\mathcal{M} \otimes \mathcal{N}}$ and $\tau_{\mathcal{N} \otimes \mathcal{M}}$.

For $\mathcal{I} \in \{\mathcal{M} \otimes \mathcal{N}, \mathcal{N} \otimes \mathcal{M}\}$ and $\mathcal{S} := \mathcal{B}(\mathbb{R}^2) * \mathcal{I}$, all conditions except for (5) can be checked by usual methods.
The fulfillment of (5) is a nontrivial result [M.B+J. Hejduk (1994/95)].
This leads to topologies $\tau_{\mathcal{M}\otimes\mathcal{N}}$ and $\tau_{\mathcal{N}\otimes\mathcal{M}}$.

Question (**still open**)

Are these topologies homeomorphic?