Universal minimal flow in the language of filters

Dana Bartošová

Trends in Set Theory, Warsaw
July 10, 2012

G a topological group

Universal minimal flow

G a topological group
X a compact Hausdorff space

Universal minimal flow

G a topological group
X a compact Hausdorff space
$\pi : G \times X \longrightarrow X$ a continuous action

Universal minimal flow

G a topological group
X a compact Hausdorff space
$\pi : G \times X \longrightarrow X$ a continuous action

Definition

X is minimal if it has no non-trivial subflow.

Universal minimal flow

G a topological group
X a compact Hausdorff space
$\pi : G \times X \longrightarrow X$ a continuous action

Definition

X is minimal if it has no non-trivial subflow.

Equivalently, X is minimal if for every $x \in X$ its orbit $Gx = \{gx : g \in G\}$ is dense in X.

Universal minimal flow

G a topological group
X a compact Hausdorff space
$\pi : G \times X \longrightarrow X$ a continuous action

Definition

X is minimal if it has no non-trivial subflow.

Equivalently, X is minimal if for every $x \in X$ its orbit $Gx = \{gx : g \in G\}$ is dense in X.

Definition

X is the universal minimal flow if every other minimal flow is its G-factor.

The universal minimal flow exists

$$\begin{array}{ccc} G \times M(G) & \xrightarrow{\;\cdot\;} & M(G) \\ {\scriptstyle \mathrm{id}\times q}\downarrow & & \downarrow{\scriptstyle \mathrm{id}\times q} \\ G \times M & \longrightarrow & M \end{array}$$

The universal minimal flow exists

$$
\begin{array}{ccc}
G \times M(G) & \xrightarrow{\quad \cdot \quad} & M(G) \\
\Big\downarrow{\scriptstyle \mathrm{id}\times q} & & \Big\downarrow{\scriptstyle \mathrm{id}\times q} \\
G \times M & \longrightarrow & M
\end{array}
$$

Theorem

The universal minimal flow exists and it is unique up to an isomorphism.

Groups of automorphisms

$\mathcal{A}$ a first order structure

Groups of automorphisms

$\mathcal{A}$ a first order structure
$G = \mathrm{Aut}(\mathcal{A})$ group of automorphisms of $\mathcal{A}$ with the topology of pointwise convergence

Groups of automorphisms

$\mathcal{A}$ a first order structure
$G = \mathrm{Aut}(\mathcal{A})$ group of automorphisms of $\mathcal{A}$ with the topology of pointwise convergence
The topology is given by clopen subgroups

Groups of automorphisms

$\mathcal{A}$ a first order structure
$G = \mathrm{Aut}(\mathcal{A})$ group of automorphisms of $\mathcal{A}$ with the topology of pointwise convergence
The topology is given by clopen subgroups

$$G_A = \{g : ga = a \ \forall a \in A\}$$

for A a finite subset of $\mathcal{A}$.

The universal minimal flow for groups of automorphisms of structures

$L = \{G_A S : S \subset G, A \subset \mathcal{A} \text{ finite}\}$ - a Boolean algebra

The universal minimal flow for groups of automorphisms of structures

$L = \{G_A S : S \subset G, A \subset \mathcal{A} \text{ finite}\}$ - a Boolean algebra

Definition

A subset $S \subset G$ is called syndetic if there exist $g_1, g_2, \ldots, g_n \in G$ such that $\cup_{i=1}^{n} g_i S = G$.

The universal minimal flow for groups of automorphisms of structures

$L = \{G_A S : S \subset G, A \subset \mathcal{A} \text{ finite}\}$ - a Boolean algebra

Definition

A subset $S \subset G$ is called syndetic *if there exist $g_1, g_2, \ldots, g_n \in G$ such that $\cup_{i=1}^n g_i S = G$.*
A subalgebra K of L is called a syndetic subalgebra *if every $G_A S \in K$ is a syndetic set and if $g \in G$ then also $gG_A S \in K$.*

The universal minimal flow for groups of automorphisms of structures

$L = \{G_A S : S \subset G, A \subset \mathcal{A} \text{ finite}\}$ - a Boolean algebra

Definition

A subset $S \subset G$ is called syndetic if there exist $g_1, g_2, \ldots, g_n \in G$ such that $\cup_{i=1}^{n} g_i S = G$.
A subalgebra K of L is called a syndetic subalgebra if every $G_A S \in K$ is a syndetic set and if $g \in G$ then also $g G_A S \in K$.

Theorem

Let $B(G)$ denote a maximal syndetic subalgebra of L with respect to inclusion. Then the universal minimal flow for $\mathrm{Aut}(\mathcal{A})$ is the Stone space of $B(G)$.

Linear orderings

$\mathrm{card}(A) = \kappa$

Linear orderings

$\mathrm{card}(A) = \kappa$

$\mathrm{LO}(\mathcal{A})$ the space of linear orderings of $\mathcal{A}$ as a subspace of $2^{\kappa \times \kappa}$

Linear orderings

$\mathrm{card}(A) = \kappa$
$\mathrm{LO}(\mathcal{A})$ the space of linear orderings of $\mathcal{A}$ as a subspace of $2^{\kappa\times\kappa}$
For $\phi \in \mathrm{Aut}(\mathcal{A})$, $<\in \mathrm{LO}(\mathcal{A})$ and $a, b \in \mathcal{A}$

$$a(\phi <)b \text{ if and only if } \phi^{-1}(\mathrm{a}) < \phi^{-1}(\mathrm{b}).$$

Linear orderings

$\mathrm{card}(A) = \kappa$

$\mathrm{LO}(\mathcal{A})$ the space of linear orderings of $\mathcal{A}$ as a subspace of $2^{\kappa\times\kappa}$

For $\phi \in \mathrm{Aut}(\mathcal{A})$, $<\in \mathrm{LO}(\mathcal{A})$ and $a, b \in \mathcal{A}$

$$a(\phi <)b \text{ if and only if } \phi^{-1}(\mathrm{a}) < \phi^{-1}(\mathrm{b}).$$

Theorem

If $\mathcal{A}$ is and infinite set or a homogeneous

- *(K_n-free) graph,*
- *(a-free) hypergraph,*
- *Boolean algebra,*
- *vector space over a finite field,*

card$(A) = \kappa$
LO$(\mathcal{A})$ the space of linear orderings of $\mathcal{A}$ as a subspace of $2^{\kappa\times\kappa}$
For $\phi \in \mathrm{Aut}(\mathcal{A})$, $<\in \mathrm{LO}(\mathcal{A})$ and $a, b \in \mathcal{A}$

$$a(\phi <)b \text{ if and only if } \phi^{-1}(\mathrm{a}) < \phi^{-1}(\mathrm{b}).$$

Theorem

If $\mathcal{A}$ is and infinite set or a homogeneous

- *(K_n-free) graph,*
- *(a-free) hypergraph,*
- *Boolean algebra,*
- *vector space over a finite field,*

then the universal minimal flow of Aut$(\mathcal{A})$ *is a space of linear orderings.*

Linear orderings

$\text{card}(A) = \kappa$
$\text{LO}(\mathcal{A})$ the space of linear orderings of $\mathcal{A}$ as a subspace of $2^{\kappa \times \kappa}$
For $\phi \in \text{Aut}(\mathcal{A})$, $< \in \text{LO}(\mathcal{A})$ and $a, b \in \mathcal{A}$

$$a(\phi <)b \text{ if and only if } \phi^{-1}(\mathrm{a}) < \phi^{-1}(\mathrm{b}).$$

Theorem

If $\mathcal{A}$ is and infinite set or a homogeneous

- *(K_n-free) graph,*
- *(a-free) hypergraph,*
- *Boolean algebra,*
- *vector space over a finite field,*

then the universal minimal flow of $\text{Aut}(\mathcal{A})$ *is a space of linear orderings.*

These results were proved in the countable case by Glasner and Weiss, Glasner and Gutman, Kechris, Pestov and Todorcevic.

Unique ergodicity

Definition

A topological group is called amenable, if every G-flow admits an invariant probability Borel measure.

Unique ergodicity

Definition

A topological group is called amenable, if every G-flow admits an invariant probability Borel measure.
If such a measure is unique, then G is uniquely ergodic.

Unique ergodicity

Definition

A topological group is called amenable, if every G-flow admits an invariant probability Borel measure.
If such a measure is unique, then G is uniquely ergodic.

Let G be

- the group of all bijection on an infinite set or
- the group of automorphisms of an infinite vector space over a finite field.

Unique ergodicity

Definition

A topological group is called amenable, if every G-flow admits an invariant probability Borel measure.
If such a measure is unique, then G is uniquely ergodic.

Let G be

- the group of all bijection on an infinite set or
- the group of automorphisms of an infinite vector space over a finite field.

Then $B(G)$ is generated by sets $G_A K$ for $A \subset G$ finite and $K \subset G$ such that for every $g \in G$ there is an $h \in G$ with

Unique ergodicity

Definition

A topological group is called amenable, if every G-flow admits an invariant probability Borel measure.
If such a measure is unique, then G is uniquely ergodic.

Let G be

- the group of all bijection on an infinite set or
- the group of automorphisms of an infinite vector space over a finite field.

Then $B(G)$ is generated by sets $G_A K$ for $A \subset G$ finite and $K \subset G$ such that for every $g \in G$ there is an $h \in G$ with

$$G_{(A)}g \cap G_A K = G_A h,$$

Definition

A topological group is called amenable, if every G-flow admits an invariant probability Borel measure.
If such a measure is unique, then G is uniquely ergodic.

Let G be

- the group of all bijection on an infinite set or
- the group of automorphisms of an infinite vector space over a finite field.

Then $B(G)$ is generated by sets $G_A K$ for $A \subset G$ finite and $K \subset G$ such that for every $g \in G$ there is an $h \in G$ with

$$G_{(A)} g \cap G_A K = G_A h,$$

where $G_{(A)} = \{g \in G : gA = A\}$.

The universal minimal flow as a space of filters

(G, τ) a topological group

The universal minimal flow as a space of filters

(G, τ) a topological group
$B(G)$ a maximal syndetic subalgebra of $\mathcal{P}(G)$

The universal minimal flow as a space of filters

(G, τ) a topological group
$B(G)$ a maximal syndetic subalgebra of $\mathcal{P}(G)$
$B(G)$ is the algebra of clopen sets of the universal minimal flow $M(G)$ for G discrete (Balcar, Franěk)

The universal minimal flow as a space of filters

(G, τ) a topological group
$B(G)$ a maximal syndetic subalgebra of $\mathcal{P}(G)$
$B(G)$ is the algebra of clopen sets of the universal minimal flow $M(G)$ for G discrete (Balcar, Franěk)

$$\begin{array}{ccc} G \times M(G) & \xrightarrow{\;\cdot\;} & M(G) \\ \Big\downarrow{\scriptstyle \mathrm{id}\times q} & & \Big\downarrow{\scriptstyle \mathrm{id}\times q} \\ G \times M(G,\tau) & \rightarrow & M(G,\tau) \end{array}$$

The universal minimal flow as a space of filters

(G, τ) a topological group
$B(G)$ a maximal syndetic subalgebra of $\mathcal{P}(G)$
$B(G)$ is the algebra of clopen sets of the universal minimal flow $M(G)$ for G discrete (Balcar, Franěk)

$$\begin{array}{ccc} G \times M(G) & \xrightarrow{\cdot} & M(G) \\ \downarrow{\scriptstyle \mathrm{id}\times q} & & \downarrow{\scriptstyle \mathrm{id}\times q} \\ G \times M(G,\tau) & \to & M(G,\tau) \end{array}$$

$\mathcal{N}$ a base of neighbourhoods of the neutral element

The universal minimal flow as a space of filters

(G, τ) a topological group
$B(G)$ a maximal syndetic subalgebra of $\mathcal{P}(G)$
$B(G)$ is the algebra of clopen sets of the universal minimal flow $M(G)$ for G discrete (Balcar, Franěk)

$$\begin{array}{ccc} G \times M(G) & \xrightarrow{\cdot} & M(G) \\ \Big\downarrow{\scriptstyle \mathrm{id}\times q} & & \Big\downarrow{\scriptstyle \mathrm{id}\times q} \\ G \times M(G,\tau) & \rightarrow & M(G,\tau) \end{array}$$

$\mathcal{N}$ a base of neighbourhoods of the neutral element
For $u \in M(G)$, let u^* be the filter on $\mathcal{P}(G)$ generated by $\{VS : V \in \mathcal{N}, S \in u\}$.

The universal minimal flow as a space of filters

(G, τ) a topological group
$B(G)$ a maximal syndetic subalgebra of $\mathcal{P}(G)$
$B(G)$ is the algebra of clopen sets of the universal minimal flow $M(G)$ for G discrete (Balcar, Franěk)

$$\begin{array}{ccc} G \times M(G) & \xrightarrow{\cdot} & M(G) \\ \Big\downarrow {\scriptstyle \mathrm{id}\times q} & & \Big\downarrow {\scriptstyle \mathrm{id}\times q} \\ G \times M(G, \tau) & \to & M(G, \tau) \end{array}$$

$\mathcal{N}$ a base of neighbourhoods of the neutral element
For $u \in M(G)$, let u^* be the filter on $\mathcal{P}(G)$ generated by $\{VS : V \in \mathcal{N}, S \in u\}$.

$$q(u) = q(v) \text{ if and only if } u^* = v^*$$

Extremely amenable groups

Definition

A topological group G is extremely amenable if its universal minimal flow is a single point.

Extremely amenable groups

Definition

A topological group G is extremely amenable if its universal minimal flow is a single point.

Theorem

G is extremely amenable if and only if for every $V \in \mathcal{N}$ and $A, B \in B(G)$

$$VA \cap VB \neq \emptyset.$$

Extremely amenable groups

Definition

A topological group G is extremely amenable if its universal minimal flow is a single point.

Theorem

G is extremely amenable if and only if for every $V \in \mathcal{N}$ and $A, B \in B(G)$

$$VA \cap VB \neq \emptyset.$$

Theorem

G is extremely amenable if and only if for every $V \in \mathcal{N}$ and $A \in B(G)$

$$VA = G,$$

it means A is dense in G.

Urysohn spaces

Urysohn spaces

$(\mathbb{U}, d)$ - an ω-homogeneous metric space containing an isometric copy of every finite metric space

Urysohn spaces

$(\mathbb{U}, d)$ - an ω-homogeneous metric space containing an isometric copy of every finite metric space
$G = \text{Iso}(U)$ - the group of isometries of $\mathbb{U}$

Urysohn spaces

$(\mathbb{U}, d)$ - an ω-homogeneous metric space containing an isometric copy of every finite metric space
$G = \mathrm{Iso}(U)$ - the group of isometries of $\mathbb{U}$
Topology generated by

$$G_X^{\varepsilon} = \{\phi \in \mathrm{Iso}(\mathbb{U}) : \forall x \in X \ d(\phi(x), x) < \varepsilon\},$$

where X is a finite subspace of $\mathbb{U}$.

Urysohn spaces

$(\mathbb{U}, d)$ - an ω-homogeneous metric space containing an isometric copy of every finite metric space

$G = \text{Iso}(U)$ - the group of isometries of $\mathbb{U}$

Topology generated by

$$G_X^\varepsilon = \{\phi \in \text{Iso}(\mathbb{U}) : \forall x \in X \ d(\phi(x), x) < \varepsilon\},$$

where X is a finite subspace of $\mathbb{U}$.

Theorem (Pestov)

$\text{Iso}(\mathbb{U})$ *is extremely amenable.*

Urysohn spaces

$(\mathbb{U}, d)$ - an ω-homogeneous metric space containing an isometric copy of every finite metric space
$G = \mathrm{Iso}(U)$ - the group of isometries of $\mathbb{U}$
Topology generated by

$$G^{\varepsilon}_X = \{\phi \in \mathrm{Iso}(\mathbb{U}) : \forall x \in X \ d(\phi(x), x) < \varepsilon\},$$

where X is a finite subspace of $\mathbb{U}$.

Theorem (Pestov)

$\mathrm{Iso}(\mathbb{U})$ *is extremely amenable.*

Proof.

Let $M(\mathrm{Iso}(\mathbb{U}))$ be the universal minimal flow for $\mathrm{Iso}(\mathbb{U})$ discrete and $A, B \in \mathrm{Cl}(M(\mathrm{Iso}(\mathbb{U})))$.

$(\mathbb{U}, d)$ - an ω-homogeneous metric space containing an isometric copy of every finite metric space
$G = \text{Iso}(U)$ - the group of isometries of $\mathbb{U}$
Topology generated by

$$G_X^\varepsilon = \{\phi \in \text{Iso}(\mathbb{U}) : \forall x \in X \ d(\phi(x), x) < \varepsilon\},$$

where X is a finite subspace of $\mathbb{U}$.

Theorem (Pestov)

$\text{Iso}(\mathbb{U})$ *is extremely amenable.*

Proof.

Let $M(\text{Iso}(\mathbb{U}))$ be the universal minimal flow for $\text{Iso}(\mathbb{U})$ discrete and $A, B \in \text{Cl}(M(\text{Iso}(\mathbb{U})))$. Then for every G_X^ε

$$G_X^\varepsilon A \cap G_X^\varepsilon B \neq \emptyset.$$

□

The end

THANK YOU FOR YOUR ATTENTION!