Settheoretical mathods in algebraic constructions

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Preliminaries

For a cardinal κ a subset E of a linear algebra A is called κ-algebrable whenever $E \cup \{0\}$ contains a κ-generated linear algebra. If E is ω-algebrable, it is simply said to be algebrable. Let us observe that the set E is κ-algebrable for $\kappa > \omega$ if and only if it contains an algebra which is a κ-dimensional linear space (see [3]). Additionally, we say that a subset E of commutative linear algebra A is strongly κ-algebrable [3] if there exists a κ-generated free algebra A' contained in $E \cup \{0\}$.

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Note, that $X = \{x_\alpha : \alpha < \kappa\} \subset E$ is a set of free generators of a free algebra $A' \subset E \cup \{0\}$ if and only if the set X' of elements of the form $x_{\alpha_1}^{k_1} x_{\alpha_2}^{k_2} ... x_{\alpha_n}^{k_n}$ is linearly independent and all linear combinations of elements from X' are in $E \cup \{0\}$.

The notions of κ-algebrability and strong κ-algebrability do not coincide. There is a simple example witnessing this: it is the subset c_{00} of c_0 consisting of all sequences with real terms equal to zero from some place. It can be proved that c_{00} is algebrable in c_0 but is not strongly 1-algebrable [3].

Most of known results on algebrability are focus on the cases between ω-algebrability and $\mathfrak{c}$-algebrabilty. Recently, $2^{\mathfrak{c}}$-algebrability was established in $\mathbb{C}^{\mathbb{C}}$ and $\mathbb{R}^{\mathbb{R}}$ using independent families of subsets of $\mathfrak{c}$ and a decomposition of $\mathbb{C}$ and $\mathbb{R}$ into $\mathfrak{c}$ copies of Bernstein sets (see [3], [2] and [1]).

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The motivation for this note comes from [2]. There was proved that the set of Zygmund-Siepiński functions is a strongly κ-algebrable subset of linear algebra $\mathbb{R}^{\mathbb{R}}$ where κ is the cardinality of a family of almost disjoint subsets of $\mathbb{R}$.

The following classical result of Kuratowski and Sierpiński [4] will be useful in the sequel.

Theorem (Disjoint Refinement Lemma)

Let $\kappa \geq \omega$. For any family $\{P_\alpha : \alpha < \kappa\}$ of sets of cardinality κ there is a family $\{Q_\alpha : \alpha < \kappa\}$ of set of cardinality κ such that

(a) $\forall \alpha < \kappa \;\; (Q_\alpha \subset P_\alpha)$;

(b) $\forall \alpha < \beta < \kappa \;\; (Q_\alpha \cap Q_\beta = \emptyset)$.

The family $\{Q_\alpha : \alpha < \kappa\}$ will be called a disjoint refinement of $\{P_\alpha : \alpha < \kappa\}$.

Theorem

Let X be a set of cardinality κ where $\kappa = \kappa^\omega$. Let I be a subset of $\mathbb{R}$ (or $\mathbb{C}$) with a non-empty interior. Then there exists a free linear subalgebra of $\mathbb{R}^X$ (or $\mathbb{C}^X$) of 2^κ generators being surjections from X to I.

Proof

At first note that if $\kappa^\omega = \kappa$, then $\kappa \geq \mathfrak{c}$. Therefore $|X| \geq |I|$, so there are surjections from X to I. Let $Y = ([0,1] \times \kappa)^{\mathbb{N}}$ and fix a family $\{A_\xi : \xi < 2^\kappa\}$ of independent subsets of κ. For each $\xi < 2^\kappa$ define $\overline{f}_\xi : Y \to [0,1]$ by the formula

$$\overline{f}_\xi(t_1, y_1, t_2, y_2, \ldots) = \prod_{n=1}^{\infty} t_n^{\chi_{A_\xi}(y_n)},$$

where $t_n \in [0,1]$, $y_n \in \kappa$, χ_{A_ξ} stands for the characteristic function of A_ξ and $0^0 = 1$.

Since $\kappa^\omega = \kappa$ and $\kappa \geq \mathfrak{c}$, then $|Y| = |X|$. Note also that I is of cardinality $\mathfrak{c}$, since it has non-empty interior. Hence we can find two bijections $\phi : X \to Y$ and $\psi : [0,1] \to I$. Then functions $f_\xi = \psi \circ \overline{f}_\xi \circ \phi$, $\xi < 2^\kappa$, are free generators in $\mathbb{R}^X$ (or in $\mathbb{C}^X$) $\square$

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We say that a map f from topological space X to $\mathbb{C}$ is strongly everywhere surjective if for every nonempty open set $U \subset X$ and every $z \in \mathbb{C}$ the cardinality of $\{x \in U : f(x) = z\}$ equals to the cardinality of U.

Theorem

The set of all strongly everywhere surjective functions $f : \beta\kappa \setminus \kappa \to \mathbb{C}$ *is strongly* $2^{2^{2^\kappa}}$*-algebrable.*

Proof.

Recall that the basis of $\beta\kappa \setminus \kappa$ consists of sets $U_A = \{p : p$ is a non-principle ultrafilter on κ such that $A \in p\}$ for $A \subset \kappa$. Every set U_A is of cardinality 2^{2^κ}. Let $\{U_\xi : \xi < 2^{2^\kappa}\}$ be an enumeration of all subsets of the basis such that each U_A appears 2^{2^κ} many times. Let $\{Q_\xi : \xi < 2^{2^\kappa}\}$ be a disjoint refinement of $\{U_\xi : \xi < 2^{2^\kappa}\}$. By Theorem 2, there is a free algebra $\mathcal{A}_\xi$ of functions of $2^{2^{2^\kappa}}$ generators $\{f^\xi_\eta : \eta < 2^{2^{2^\kappa}}\}$ being surjections from Q_ξ onto $\mathbb{C}$. Since polynomials are surjections from $\mathbb{C}^n$ onto $\mathbb{C}$, then $\mathcal{A}_\xi$ consists of surjections from Q_ξ onto $\mathbb{C}$.

Let $\mathcal{A}$ be the algebra generated by $\{f_\eta : \eta < 2^{2^{2^\kappa}}\}$ where $f_\eta(x) = f^\xi_\eta(x)$ if $x \in Q_\xi$, and $f_\eta(x) = 0$ if $x \notin \bigcup_{\xi < 2^{2^\kappa}} Q_\xi$. Then $\mathcal{A}$ is clearly a free linear algebra consisting of strongly everywhere surjective functions from $\beta\kappa \setminus \kappa$ to $\mathbb{C}$. □

A function $f : \mathbb{C} \to \mathbb{C}$ is called *strongly everywhere surjective* if f takes $\mathfrak{c}$ many times every value $z \in \mathbb{C}$ on every non-empty open subset of $\mathbb{C}$. A function $f : \mathbb{C} \to \mathbb{C}$ is called *perfectly everywhere surjective* if f maps every perfect subset of $\mathbb{C}$ onto $\mathbb{C}$. We will denote the above classes of functions by $\mathcal{SES}(\mathbb{C})$ and $\mathcal{PES}(\mathbb{C})$.

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By $\mathcal{EDD}(\mathbb{R})$ we denote the set of all everywhere discontinuous Darboux functions (from $\mathbb{R}$ to $\mathbb{R}$), i.e. nowhere continuous functions which map connected sets onto connected sets.

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A function $f : \mathbb{R} \to \mathbb{R}$ is called *Sierpiński function* if any perfect set $P \subset \mathbb{R}$ contains a perfect set $Q \subset P$ such that $f|_Q$ is continuous.

Theorem

The following families of functions are strongly $2^{\mathfrak{c}}$-algebrable:

(i) $\mathcal{PES}(\mathbb{C})$;

(ii) $\mathcal{SES}(\mathbb{C}) \setminus \mathcal{PES}(\mathbb{C})$;

(iii) $\mathcal{EDD}(\mathbb{R})$;

(iv) *functions whose sets of continuity points equal K for a fixed closed set $K \subsetneq \mathbb{R}$ (or $K \subsetneq \mathbb{C}$).*

Theorem

By $\mathcal{F}$ denote the set of all functions $f : \mathbb{C} \to \mathbb{C}$ which fulfils the following conditions:

(a) *f is strongly everywhere surjective;*

(b) *for any perfect set S there is a perfect set $S' \subset S$ with $f|_{S'} = 0$, in particular f is $s(\mathbb{C})$-measurable;*

(c) *f is Lebesgue measurable and f has the Baire property.*

Then $\mathcal{F}$ is strongly $2^{\mathfrak{c}}$-algebrable.

Let $\mathcal{F}_1$ be the family of all functions from $\mathcal{SES}(\mathbb{C})$ which have the Baire property but are neither measurable nor s-measurable. Let $\mathcal{F}_2$ be the family of all functions from $\mathcal{SES}(\mathbb{C})$ which are measurable but neither have the Baire property nor are s-measurable.

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Theorem

The families $\mathcal{F}_1$ and $\mathcal{F}_2$ are strongly $2^{\mathfrak{c}}$-algebrable.

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