Topologies defined on trees

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For a partial order $(X, \leq)$ and $y \in X$ we shall use the following abbreviations:

$$(\leftarrow, y) = \{z \in X \colon z < y\},$$

$$(y, \rightarrow) = \{z \in X \colon z > y\},$$

A tree is a partial order $(T, \leq)$ such that:

1. there exists the least element in T,

2. for every $t \in T$ the set $(\leftarrow, t)$ is well ordered.

The order type of the set $(\leftarrow, t)$ is called the height of t in T and denoted by $\mathrm{ht}(t, T)$ whereas

$$\mathrm{Lev}_\alpha(T) = \{t \in T \colon \mathrm{ht}(t, T) = \alpha\}$$

is the α–level of T.

The height of the tree $(T, \leq)$ is defined as

$$\text{ht}(T) = \min\{\alpha\colon \text{Lev}_\alpha(T) = \emptyset\}.$$

If $(T, \leq)$ is a tree and $t \in T$ then

$$\text{succ}(t) = \{s \in T\colon s \text{ is minimal in } (t, \rightarrow)\}$$

denotes the set of all immediate successors of the element t.

A tree $(T, \leq)$ is called infinitely branching whenever the set succ(t) is infinite for every $t \in T$.

A family $\mathcal{F} \subseteq \mathcal{P}(X)$ is called a (free) filter whenever

1. $\{X \setminus F\colon |F| < \omega\} \subseteq \mathcal{F}$ and $\emptyset \notin \mathcal{F}$,

2. $(\forall F \in \mathcal{F})(\forall G \subseteq X)(F \subseteq G \Rightarrow G \in \mathcal{F})$,

3. $(\forall F_1, F_2 \in \mathcal{F})(F_1 \cap F_2 \in \mathcal{F})$.

Let $(T, \leq)$ be a tree and let $\mathfrak{F} = (\mathcal{F}_t \colon t \in T)$, where

$$\mathcal{F}_t \subseteq \mathcal{P}(\mathrm{succ}(t))$$

for every $t \in T$, be an indexed family of filters. For every $s \in T$ and every

$$\phi_s \in \prod\{\mathcal{F}_t \colon t \in [s, \to)\},$$

we consider the set

$$U_{s,\phi_s} = \bigcup\{U^\alpha_{\phi_s} \colon \alpha < \mathrm{ht}[s, \to)\},$$

where for every $\alpha < \mathrm{ht}[s, \to)$ the sets $U^\alpha_{\phi_s} \subseteq T$ are defined as follows:

$$U^0_{\phi_s} = \{s\},$$

$$U^{\alpha+1}_{\phi_s} = U^\alpha_{\phi_s} \cup \bigcup\{\phi_s(t) \colon t \in U^\alpha_{\phi_s} \text{ and } \mathrm{ht}(t, [s, \to)) = \alpha\},$$

$$U^\alpha_{\phi_s} = \{t \in T \colon [s, t) \subseteq \bigcup\{U^\beta_{\phi_s} \colon \beta < \alpha\}\}$$

if α is a limit ordinal.

For every tree T and every indexed family $\mathfrak{F} = (\mathcal{F}_t \colon t \in T)$ of filters we consider the collection

$$\mathcal{B}(T, \mathfrak{F}) = \{U_{s,\phi_s} \colon s \in T \text{ and } \phi_s \in \prod\{\mathcal{F}_t \colon t \in [s, \to)\}\}.$$

Lemma *The family $\mathcal{B}(T, \mathfrak{F}) \cup \{\emptyset\}$ is closed under finite intersections.*

Definition 1 *The* tree topology *$\mathcal{T}_{\mathfrak{F}}$ on T is the topology generated by the family $\mathcal{B}(T, \mathfrak{F})$, where $\mathfrak{F} = (\mathcal{F}_t \colon t \in T)$ is an indexed family of filters. A tree endowed with the tree topology $\mathcal{T}_{\mathfrak{F}}$ is called an* $\underline{\mathfrak{F}\text{–tree.}}$

Theorem *Let $(T, \leq)$ be an $\mathfrak{F}$–tree of height $\kappa \geq \omega$. Then the following conditions hold true:*

(1) T is a zero–dimensional dense in itself Hausdorff space,

(2) T is nowhere compact, i.e. if $A \subseteq T$ is a compact subspace then $\operatorname{int} A = \emptyset$,

(3) $\operatorname{cl}\mathsf{Lev}_\alpha(T) = \mathsf{Lev}_{\leq\alpha}(T)$ *for every* $\alpha < \kappa$,

(4) $\operatorname{int}\mathsf{Lev}_{\leq\alpha}(T) = \emptyset$ *for every* $\alpha < \kappa$,

(5) if $A \subseteq T$ *is a chain, then* A *is closed and discrete,*

(6) if $A \subseteq T$ *is an antichain, then* A *is a discrete subspace of* T*.*

Proposition *Every countable* $\mathfrak{F}$*–tree has a continuous bijection onto the space of rational numbers.*

Proposition *Let* $(T, \leq)$ *be a special Aronszajn tree. Then for every* $\mathfrak{F} = (\mathcal{F}_t\colon t \in T)$ *the* $\mathfrak{F}$*–tree* T *is an uncountable dense in itself space which is a countable union of closed discrete subspaces.*

Theorem *Every $\mathfrak{F}$-tree is a collectionwise normal space.*

Theorem *Assume T is an $\mathfrak{F}$–tree with $\mathfrak{F} = (\mathcal{F}_t \colon t \in T)$ and $\mathrm{ht}(T) = \omega$. Then T is extremally disconnected iff for every $t \in T$ the filter $\mathcal{F}_t$ is an ultrafilter.*

Proposition *Let $(T, \leq)$ be a tree with the underlying set*

$$T = \mathrm{Seq}(\omega + \omega) = \bigcup \{ {}^{\alpha}\omega : \alpha < \omega + \omega \}$$

and the partial order given by

$$x \leq y \Longleftrightarrow y \restriction \mathrm{dom}(x) = x,$$

and let $\mathfrak{F} = (\mathcal{F}_t \colon t \in T)$ be an arbitrary collection of filters. Then there exist disjoint sets $U, V \subseteq T$ which are open in the $\mathfrak{F}$–tree T and such that $\emptyset \in \mathrm{cl}\, U \cap \mathrm{cl}\, V$. In particular, the $\mathfrak{F}$–tree T is not extremally disconnected.

Later we shall assume additionally that ht$(T) = \omega$ and there exists a cardinal $\kappa \geq \omega$ such that $|\,\mathrm{succ}(s)| = \kappa$ for all $s \in T$. Hence, the $\mathfrak{F}$-tree T is extremally disconnected whenever $\mathfrak{F}$ consist of ultrafilters.

Theorem *Assume $(T, \leq)$ is an $\mathfrak{F}$-tree and let $f\colon T \to T$ be a continuous closed mapping. If $\mathfrak{F}$ consists of pairwise incomparable ultrafilters, then f is the identity.*

Theorem *Assume $(T, \leq)$ is an $\mathfrak{F}$-tree and $\mathfrak{F}$ consists of pairwise incomparable ultrafilters. If $U, V \subseteq T$ are open sets and $f\colon U \to V$ is an open surjection, then $U = V$ and f is the identity.*

Remark **(Jerry Vaugran)** *Assume $T =$ Seq is an $\mathfrak{F}$–tree where $\mathfrak{F} = (\mathcal{F}_s\colon s \in \text{Seq})$ is a collection of pairwise comparable ultrafilters. Then every two nonempty open subsets of T are homeomorphic.*

Theorem *Assume* $T = \mathrm{Seq}$ *is an* $\mathfrak{F}$*–tree where* $\mathfrak{F} = (\mathcal{F}_s \colon s \in \mathrm{Seq})$ *is a collection of pairwise incomparable weak P–ultrafilters. Then for every continuous injection* $f \colon \beta T \to \beta T$ *there exists a clopen set* $U \subseteq \beta T$ *such that* $f \restriction U$ *is the identity and* $f[\beta T \setminus U]$ *is a nowhere dense subset of* βT*. In particular, if* f *is a homeomorphism of* βT *onto itself, then it is the identity.*

If $\lambda > \omega$, then a (free) filter $\mathcal{F}$ on ω is called a $\underline{P_\lambda\text{-filter}}$ if for every subfamily $\mathcal{F}$ of size less than λ there exists an element of $\mathcal{F}$ which is almost contained in every element of $\mathcal{F}$. As usual $\mathfrak{b}$ denotes the minimal cardinality of an unbounded subset of ${}^\omega\omega$ ordered by the relation $\leq^*$.

Theorem *Assume* $\omega < \lambda \leq \mathfrak{b}$ *and* $T = \mathrm{Seq}(\omega)$*. If* $\mathfrak{F} = (\mathcal{F}_t \colon t \in T)$ *consists of* P_λ*-filters and* $\mathcal{U}$ *is a collection of open subsets of* βT *such that* $|\mathcal{U}| < \lambda$ *and* $T \subseteq U \subseteq \beta T$ *for every* $U \in \mathcal{U}$*, then*

$$T \subseteq \mathrm{int} \bigcap \mathcal{U}.$$