Recent results on splitting and almost disjointness

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1. Closed almost disjointness
2. Tail splitting

Mathias' Theorem

$\mathcal{A} \subseteq [\omega]^\omega$ *a.d. family*: $|A \cap B| < \omega$ for $A \neq B \in \mathcal{A}$
$\mathcal{A}$ *mad family*: $\mathcal{A}$ is a.d. and maximal
(i.e., for all $C \in [\omega]^\omega$ there is $A \in \mathcal{A}$ with $|C \cap A| = \omega$.)

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$\mathfrak{a}_{\mathrm{Borel}} := \min\{|\mathcal{B}| : \mathcal{B}$ infinite family of a.d. Borel sets, $\bigcup \mathcal{B}$ mad$\}$,
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$\mathfrak{a}_{\mathrm{closed}} := \min\{|\mathcal{B}| : \mathcal{B}$ infinite family of a.d. closed sets, $\bigcup \mathcal{B}$ mad$\}$, the *closed almost disjointness number*.

Review: some cardinal invariants 1

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We recall some definitions.

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$\mathcal{T} = \{T_\alpha : \alpha < \kappa\} \subseteq [\omega]^\omega$ is a *tower* if

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Let $\mathbb{P}$ be a forcing notion.

$\mathfrak{h}(\mathbb{P}) := \min\{|\mathcal{D}| : \mathcal{D}$ is a family of open dense sets in $\mathbb{P}$ and $\bigcap \mathcal{D}$ is not dense$\}$, the *distributivity number of* $\mathbb{P}$.

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$\mathfrak{a}_{\text{Borel}} \geq \mathfrak{t}$.

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$\mathcal{A}$ mad family. Then:
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Does $\mathfrak{a}_{\mathrm{closed}} = \mathfrak{a}_{\mathrm{Borel}}$?

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What about the relationship between $\mathfrak{a}$ and $\mathfrak{a}_{\text{closed}}$?

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Proof Sketch. A forcing notion $\mathbb{P}$ is *almost ω^ω-bounding* if for all $p \in \mathbb{P}$ and $\dot{f} \in \omega^\omega$ there are $q \leq p$ and $g \in \omega^\omega$ such that $\forall X \in [\omega]^\omega \;\; \exists q_X \leq q$ with $q_X \Vdash \exists^\infty n(\dot{f}(n) \leq g(n))$.

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Shelah constructed a creature forcing $\mathbb{P}$ for forcing $\mathfrak{s} > \mathfrak{b}$. $\mathbb{P}$ is proper and almost ω^ω-bounding and adds an unsplit real.

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$\mathfrak{a}_{\mathrm{closed}}$ versus $\mathfrak{b}$ 3

Now, given a family of closed sets $\mathcal{B}$ such that $\bigcup \mathcal{B}$ is a mad family, one can define a forcing $\mathbb{P}_{\mathcal{B}}$ such that after forcing with $\mathbb{P}_{\mathcal{B}}$, the *reinterpretation* of $\bigcup \mathcal{B}$ is not mad anymore.

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Let $\mathbb{F}$ be the forcing notion consisting of all F_σ-filters conversely ordered by inclusion. $\mathbb{F}$ generically adds a P-point $\dot{\mathcal{U}}$.

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Theorem (Raghavan)

$\mathbb{P} \cong \mathbb{F} \star \mathbb{M}(\dot{\mathcal{U}})$. *That is,* $\mathbb{P}$ *is forcing equivalent to the two-step iteration of* $\mathbb{F}$ *and Mathias forcing* $\mathbb{M}(\dot{\mathcal{U}})$ *with the generic P-point* $\dot{\mathcal{U}}$.

1. Closed almost disjointness

2. Tail splitting

Tail splitting 1

Suppose $\mathfrak{b} = \aleph_1$. Then there is $\mathcal{F} = \{f_\alpha : \alpha < \omega_1\} \subseteq \omega^\omega$ such that for all $g \in \omega^\omega$ there is α such that $f_\beta \not\leq^* g$ for all $\beta \geq \alpha$. ("$\mathcal{F}$ is unbounded on a tail.")

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A sequence $\bar{A} = (A_\alpha : \alpha < \omega_1) \subseteq [\omega]^\omega$ is *tail-splitting* if for all $B \in [\omega]^\omega$ there is α such that A_β splits B for all $\beta \geq \alpha$.

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$\bar{A} = (A_\alpha : \alpha < \omega_1)$ is *club-splitting* if for all $B \in [\omega]^\omega$, the set $\{\alpha < \omega_1 : A_\alpha \text{ splits } B\}$ contains a club.

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Suppose $\mathfrak{b} = \aleph_1$. Then there is $\mathcal{F} = \{f_\alpha : \alpha < \omega_1\} \subseteq \omega^\omega$ such that for all $g \in \omega^\omega$ there is α such that $f_\beta \not\leq^* g$ for all $\beta \geq \alpha$. ("$\mathcal{F}$ is unbounded on a tail.")

Is there a similar phenomenon for splitting?

A sequence $\bar{A} = (A_\alpha : \alpha < \omega_1) \subseteq [\omega]^\omega$ is *tail-splitting* if for all $B \in [\omega]^\omega$ there is α such that A_β splits B for all $\beta \geq \alpha$.

$\bar{A} = (A_\alpha : \alpha < \omega_1)$ is *club-splitting* if for all $B \in [\omega]^\omega$, the set $\{\alpha < \omega_1 : A_\alpha \text{ splits } B\}$ contains a club.

$\mathfrak{s}_\omega := \min\{|\mathcal{F}| : \forall\, (B_n : n \in \omega)\ \exists\, A \in \mathcal{F} \text{ splitting all } B_n\}$,
the *ω-splitting number*.

Tail splitting 2

Clear: $\exists$ tail-splitting sequence $\Longrightarrow \exists$ club-splitting sequence
$\Longrightarrow \mathfrak{s}_\omega = \aleph_1 \Longrightarrow \mathfrak{s} = \aleph_1$

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Proof Sketch. Note that "$\bar{A}$ is club-splitting" is preserved in limit stages of finite support iterations.

Tail splitting 3

Let $\bar{A}$ be club-splitting, and let $\mathcal{F}$ be a filter on ω.

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$(\star)_{\bar{A},\mathcal{F}}$ means:
for all partial $f : \omega \to \omega$, $\mathrm{dom}(f) \in \mathcal{F}^+$, $f^{-1}(\{n\}) \in \mathcal{F}^*$ $(n \in \omega)$:
$D_f = \{\alpha < \omega_1 : f^{-1}(A_\alpha), f^{-1}(\omega \setminus A_\alpha) \in \mathcal{F}^+\}$ contains a club.

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Order: inclusion $T \leq S \iff T \subseteq S$.

$\mathbb{L}(\mathcal{F})$ is a σ-centered forcing notion diagonalizing the filter $\mathcal{F}$ (and adding a dominating real).

Tail splitting 4

Lemma

Assume $(\star)_{\bar{A},\mathcal{F}}$. *Then* $\mathbb{L}(\mathcal{F})$ *preserves* $\bar{A}$.

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The proof is a standard rank argument.

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(CH) Assume $\bar{B} = (B_\alpha : \alpha < \omega_1)$ *is tail-splitting. Then there is* $(F_\alpha : \alpha < \omega_1)$ *generating a P-filter* $\mathcal{F}$ *such that* $(\star)_{\bar{A},\mathcal{F}}$ *holds and* $F_\alpha = B_{\xi_\alpha}$ *for some* $\xi_\alpha \geq \alpha$.

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The proof is technical but straightforward.

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Lemma

Assume $(\star)_{\bar{A},\mathcal{F}}$. Then $\mathbb{L}(\mathcal{F})$ preserves $\bar{A}$.

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(CH) Assume $\bar{B} = (B_\alpha : \alpha < \omega_1)$ is tail-splitting. Then there is $(F_\alpha : \alpha < \omega_1)$ generating a P-filter $\mathcal{F}$ such that $(\star)_{\bar{A},\mathcal{F}}$ holds and $F_\alpha = B_{\xi_\alpha}$ for some $\xi_\alpha \geq \alpha$.

Since $\mathbb{L}(\mathcal{F})$ diagonalizes $\mathcal{F}$, $\bar{B}$ is not tail-splitting anymore after forcing with $\mathbb{L}(\mathcal{F})$.

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Since $\mathbb{L}(\mathcal{F})$ diagonalizes $\mathcal{F}$, $\bar{B}$ is not tail-splitting anymore after forcing with $\mathbb{L}(\mathcal{F})$. This completes the proof of the theorem.

Tail splitting and closed almost disjointness 1

$\bar{A} = (A_{\alpha,n} : \alpha < \omega_1, n < \omega) \subseteq [\omega]^\omega$ is a *tail-splitting sequence of partitions* if the $A_{\alpha,n}$ $(n \in \omega)$ are pairwise disjoint and for all $B \in [\omega]^\omega$ there is α such that $A_{\beta,n}$ splits B for all $\beta \geq \alpha$ and all $n \in \omega$.

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So this is slightly stronger than a tail-splitting sequence.

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Theorem

The existence of a tail-splitting sequence of partitions implies $\mathfrak{a}_{\text{closed}} = \aleph_1$.

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Theorem

The existence of a tail-splitting sequence of partitions implies $\mathfrak{a}_{\text{closed}} = \aleph_1$.

This can be distilled from the proof of CON($\mathfrak{a}_{\text{closed}} < \mathfrak{b}$).

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There is a tail-splitting sequence of partitions in the Hechler model.

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$\mathfrak{d} = \aleph_1$ *implies the existence of a tail-splitting sequence of partitions.*

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$\mathfrak{d} = \aleph_1$ *implies the existence of a tail-splitting sequence of partitions.*

Thus both the Raghavan-Shelah Theorem and CON($\mathfrak{a}_{\text{closed}} < \mathfrak{b}$) follow from this theorem.

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Thus both the Raghavan-Shelah Theorem and CON($\mathfrak{a}_{\text{closed}} < \mathfrak{b}$) follow from this theorem.

Question

Does $\mathfrak{s} = \aleph_1$ imply that $\mathfrak{a}_{\text{closed}} = \aleph_1$?