Grigorieff forcing and automorphisms of $\mathcal{P}(\omega)/\operatorname{Fin}$

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Let φ be an automorphism of the Boolean algebra $\mathcal{P}(\omega)/\operatorname{Fin}$

A partial 1-1 function $f\colon A \to A$ is an almost permutation if $\operatorname{Dom}(f) =^* A =^* \operatorname{Rng}(f)$

Each bijection $f\colon A \to B$ induces a homeomorphism $\psi\colon \mathcal{P}(A)/\operatorname{Fin} \to \mathcal{P}(B)/\operatorname{Fin}$ by $\psi([C]) = [f[C]]$ for $C \in \mathcal{P}(A)$

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- trivial $\Leftrightarrow$ trivial on $\{\omega\}$
- somewhere trivial $\Leftrightarrow$ trivial on $[\omega]^\omega$

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  Generally not possible!
- Option 2: Killing φ – add new subsets of ω so that φ cannot be extended to an automorphism in the extension.

Definition (Grigorieff's forcing)

Let $\mathcal{F}$ be a filter on ω. Put

$$G(\mathcal{F}) = \{p \colon I \to 2; I \in \mathcal{F}^*\}$$

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Theorem (Ch., Dow)

Let φ be an automorphism of $\mathcal{P}(\omega)/\operatorname{Fin}$ and let $\mathcal{F}$ be a non-meager p-filter such that φ is not trivial on $\mathcal{F}$.
Let g be the $G(\mathcal{F})$-generic real.
The family

$$\langle \varphi(p^{-1}(1)), \varphi(p^{-1}(0))\colon p \in g\rangle$$

is an unfilled gap (in $V[g]$).

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Theorem (Abraham, Todorcevic)

(GCH) *For each (ω_1, ω_1) gap $\mathcal{A}$ there exist a proper ${}^{\omega}\omega$ bounding (not adding new reals) ω_2-p.i.c. forcing which makes $\mathcal{A}$ indestructible in the extension.*

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Corollary

(GCH) *Let φ be an automorphism which is not trivial on a non-meager p-filter $\mathcal{F}$.*
There is a proper ${}^{\omega}\omega$ bounding ω_2-p.i.c. forcing $\mathbb{P}$ such that there is no automorphism extending φ in any ω_1 preserving extension of $V[G_{\mathbb{P}}]$.

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Corollary

It is consistent with ZFC *that* $\mathfrak{d} = \omega_1$ *and every automorphisms of* $\mathcal{P}(\omega)/\operatorname{Fin}$ *is trivial on each non-meager p-filter.*

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Let φ be an automorphism of $\mathcal{P}(\omega)/\operatorname{Fin}$ and let $\mathcal{F}$ be a non-meager p-filter such that φ is not trivial on $\mathcal{F}$. The family

$$\mathcal{A} = \langle \varphi(p^{-1}(1)), \varphi(p^{-1}(0)) \colon p \in g \rangle$$

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Proof.

Let $\dot{y}$ be a $G(\mathcal{F})$ name for a real (a candidate for filling $\mathcal{A}$).
Fix a countable elementary submodel M containing $\dot{y}$.
M-generic condition q forces that $\dot{y}$ looks like a Cohen name.
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Lemma

Let φ be a non-trivial automorphism and $\mathcal{I}$ be a non-meager p-ideal. Suppose $F\colon \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ is function continuous on a dense G_δ set.
There are $x \subset a \in \mathcal{I}$ such that

$$\mathbf{C} \Vdash F(v) \cap \varphi(a) \neq^* \varphi(x) \text{ for each } v =^* x \cup g_{\omega \setminus a}$$

where $\mathbf{C}$ is Cohen forcing and $g_{\omega\setminus a}$ is Cohen generic subset of $\omega \setminus a$.