Separable reduction theorems by the method of elementary submodels

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Attempt to extend the validity of results proven in separable spaces into the nonseparable setting without knowing the proof in separable spaces. Therefore, trying to see whether some properties of sets and functions are separably determined.

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- A is meager in X if and only if $A \cap X_M$ is meager in X_M.
- For every $a \in X_M$ it is true that f is continuous at a if and only if $f \restriction_{X_M}$ is continuous at a.

Countable models theorem

Recall:

- Let M be a fixed set and φ a formula. Then φ^M is a formula which is obtained from φ by replacing each quantifier of the form "$\forall x$" by "$\forall x \in M$" and each quantifier of the form "$\exists x$" by "$\exists x \in M$".

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- Formula $\varphi(x_1, \ldots, x_n)$ is absolute for M, if for every $a_1, \ldots, a_n \in M$ holds:

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Theorem (countable models)

Let $\varphi_1, \dots, \varphi_n$ be any formulas. Then for every countable set Y there exists a countable set $M \supset Y$ such that $\varphi_1, \dots, \varphi_n$ are absolute for M.

Elementary submodels

Convention:

Whenever we say

for a suitable elementary submodel M (the following holds...),

we mean by this

there exists a list of formulas $\varphi_1, \dots, \varphi_n$ and a countable set Y such that for every countable set $M \supset Y$ such that $\varphi_1, \dots, \varphi_n$ are absolute for M (the following holds...).

The strucutre of elementary submodels

Example - structure of the models:

$\varphi_1(x,a) := \forall z(z \in x \iff ((z \in a) \vee (z = a)))$ $\quad [x = a \cup \{a\}]$
$\varphi_2(a) := \exists x \varphi_1(x,a)$
Let M be countable set such that φ_1, φ_2 are absolute for M. Then $a \cup \{a\} \in M$ whenever $a \in M$.

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- Absoluteness $\Rightarrow \exists x \in M \varphi_1^M(x, a)$. Fix such $x \in M$.
- $\varphi_1^M(x, a)$ holds, so (using the absoluteness of φ_1) $\varphi_1(x, a)$ holds as well.
- The only possibility: $a \cup \{a\} = x$; hence, $a \cup \{a\} \in M$.

□

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The usage of the technic of elementary submodels

- X normed linear space, then $X_M := \overline{X \cap M}$ is a closed separable subspace
- **We want to prove:** X normed linear space, $A \subset X$. Then there exists closed separable subspace $X_M \subset X$ such that A is residual in X if and only if $A \cap X_M$ is residual in X_M.
- **It is sufficient:** For a suitable elementary submodel M it is true that whenever M contains X and A, then A is residual in X if and only if $A \cap X_M$ is residual in X_M.

Advantaged of the method of elementary submodels

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- Part of the problem is hidden in the Countable models theorem. Therefore, the situation is less complicated and propositions are more easy to prove.
- It is possible to combine finite many results together.
- An elementary submodel M doesn't depend on the space X, so there is a chance to see some connection among various spaces (for example if $X = \ell_p(\Gamma)$, then X_M can be identified with the space $\ell_p(\Gamma \cap M)$).

Results concerning set properties

For example the following theorem holds:

Theorem

For a suitable elementary submodel M the following holds:
Let $\langle X, \rho \rangle$ be a complete metric space and $A \subset X$ a Borel set. Then whenever M contains X and A, it is true that

$$A \text{ is } \begin{array}{c}\textbf{dense}\\ \textbf{nowhere dense}\\ \textbf{meager}\\ \textbf{residual}\\ \sigma\textbf{-lower porous}\\ \sigma\textbf{-upper porous}\end{array} \text{ in } X \iff A \cap X_M \text{ is } \begin{array}{c}\textbf{dense}\\ \textbf{nowhere dense}\\ \textbf{meager}\\ \textbf{residual}\\ \sigma\textbf{-lower porous}\\ \sigma\textbf{-upper porous}\end{array} \text{ in } X_M.$$

Results concerning function properties

- **Separably determined function properties in normed linear spaces:**

  If X is a normed linear space, f a mapping, M a suitable elementary submodel and $x \in X_M$, then:

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- **Separably determined function properties in normed linear spaces:**

  If X is a normed linear space, f a mapping, M a suitable elementary submodel and $x \in X_M$, then:

  - f is continuous at $x \iff f\restriction_{X_M}$ is continuous at x.
  - f is Fréchet differentiable at $x \iff f\restriction_{X_M}$ is Fréchet differentiable at x.

Results concerning function properties

By combining the results, for example the following theorem holds

$C(f)$ (resp. $D(f)$) stands for the set of points where a function f is continuous (resp. Fréchet differentiable):

Theorem

For a suitable elementary submodel M the following holds: Let X, Y be Banach spaces and $f : X \to Y$ a function. Then whenever M contains X, Y and f, it is true that:

$$\begin{matrix} C(f) \\ D(f) \end{matrix} \text{ is } \begin{matrix} \text{dense} \\ \text{residual} \end{matrix} \text{ in } X \iff \begin{matrix} C(f\restriction_{X_M}) \\ D(f\restriction_{X_M}) \end{matrix} \text{ is } \begin{matrix} \text{dense} \\ \text{residual} \end{matrix} \text{ in } X_M.$$

From spaces with separable dual to general Asplund space

- **L.Zajíček:** For a Banach space with a separable dual holds (under certain assumptions) that the set $D(f)$ is residual.

From spaces with separable dual to general Asplund space

The exact formulation of the theorem:

Let $X = X_1 \oplus \ldots \oplus X_n$ be a Banach space with a separable dual X^*. Let $G \subset X$ be an open set and $f : G \to \mathbb{R}$ a locally Lipschitz function. Let, for each $1 \leq i \leq n$, there exists a dense set $D_i \subset S_{X_i}$ such that, for each $v \in D_i$, f is essentially smooth on a generic line parallel to v. Then $D(f)$ is residual in G.

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- **L.Zajíček:** For a Banach space with a separable dual holds (under certain assumptions) that the set $D(f)$ is residual.
- **Separable reduction:** Under the same assumptions holds even in a general Asplund space.

From $\mathcal{C}(K)$ with countable compact K to $\mathcal{C}(K)$ with a general scattered compact K

- **J.Lindenstrauss + D.Preiss:** The following spaces have the property that every Lipchitz mapping of them into space with the RNP is Fréchet differentiable everywhere except a Γ-null set: $\mathcal{C}(K)$ for countable compact K, subspaces of c_0.

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  $\mathcal{C}(K)$ is the set of continuous functions $f : K \longrightarrow \mathbb{R}$

- **Separable reduction:** Under the same assumptions holds even for spaces $\mathcal{C}(K)$ with a general scattered compact K and for subspaces of $c_0(\Gamma)$ with an arbitrary set Γ.

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The end

Thank you for your attention!