Borel equivalence relations and the Laver forcing

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Introduction

Vladimir Kanovei, Marcin Sabok and Jindřich Zapletal in *Canonical Ramsey theory on Polish spaces* deals in general with the following problem:

Let X be a Polish space, $I \subseteq \mathcal{P}(X)$ a σ-ideal on X and $E \in \mathrm{Borel}(X \times X)$ an equivalence relation.

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Next we are given a Borel set $B \in I^+$ and we ask whether there exists an I-positive Borel subset $C \subseteq B$ such that $E \restriction C <_B E \restriction B$.

Introduction-Spectrum of a σ-ideal

If there exists a Borel set $B \in I^+$ such that $\forall C \in (I^+ \cap \mathrm{Borel}(B))$ $E \restriction C$ has the same complexity as E on the whole space, i.e. $E \restriction C \approx_B E \restriction X$, then we say that E is in the spectrum of I.

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On the other hand, E can be canonized to a relation $F \leq_{\mathrm{B}} E$ if for every Borel $B \in I^+$ there is a subset $C \in (I^+ \cap \mathrm{Borel}(B))$ such that $E \restriction C \approx_B F \restriction C$.

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There is a σ-ideal I on ω^ω such that $\mathrm{Borel}(\omega^\omega) \setminus I$ is forcing equivalent to the Laver forcing.
In fact, for every analytic set $A \subseteq \omega^\omega$, either $A \in I$ or there exists a Laver tree T such that $[T] \subseteq A$.

Spectrum of Laver

The following theorem is proved in the book of Kanovei, Sabok and Zapletal, *Canonical Ramsey theory on Polish spaces*:

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Let I be a σ-ideal on a Polish space X such that the quotient forcing P_I is proper, nowhere ccc and adds a minimal forcing extension. Then I has total canonization for equivalence relations classifiable by countable structures.

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Corollary

Let T be a Laver tree, E an equivalence classifiable by countable structures. Then there is a Laver subtree on which E is either the identity relation or the full relation.

Spectrum of Laver

J. Zapletal found the following F_σ equivalence relation (denoted here as) K on ω^ω (with K_σ classes) which is in the spectrum of Laver.

We set xKy iff
$\exists b \forall m \exists n_x, n_y \leq b(x(m) \leq y(m+n_y) \wedge y(m) \leq x(m+n_x))$.

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[KaSaZa] For any Laver tree T, $K \restriction [T] \approx_B K$.

Also, for any two Laver trees T and S there are $x_0, x_1 \in [T]$ and $y_0, y_1 \in [S]$ such that $x_0 K y_0$ and $x_1 \not K y_1$.

Borel equivalences we will work with

Definition

Let $\mathcal{I}$ be a Borel ideal on ω. It induces a Borel equivalence relation $E_{\mathcal{I}}$ (of the same complexity) on 2^ω defined as:
$xE_{\mathcal{I}}y \equiv \{n \in \omega : x(n) \neq y(n)\} \in \mathcal{I}$.

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We will consider the equivalences E_{ℓ_p} for $p \in [1, \infty]$; so $xE_{\ell_p}y$ if $x - y \in \ell_p$, i.e.

- $\sum_{i=0}^{\infty}(x(i) - y(i))^p < \infty$, for $p \in [1, \infty)$
- $\{x(i) - y(i) : i \in \omega\}$ is bounded, for $p = \infty$

Main theorem

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Let T be a Laver tree, $\mathcal{I}$ an F_σ P-ideal on ω and E an equivalence relation on $[T]$ that is Borel reducible to $E_{\mathcal{I}}$. Then there is a Laver subtree $S \leq T$ such that $E \restriction [S]$ is either $\mathrm{id}([S])$ or $[S] \times [S]$.

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Corollary

In particular, for a Laver tree T, E an equivalence on $[T]$ that is Borel reducible to E_2 or E_{ℓ_p} for $p \in [1, \infty)$, there is a Laver subtree $S \leq T$ such that $E \restriction [S]$ is either $\mathrm{id}([S])$ or $[S] \times [S]$.

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Proof.

The relation K defined before is Borel bireducible with E_{ℓ_∞} (and with E_{K_σ}). □

Corollaries

Theorem

Let $E \subseteq \omega^\omega \times \omega^\omega$ *be an equivalence relation containing* K*, i.e.* $E \supseteq K$*, which is Borel reducible to* $E_{\mathcal{I}}$ *for some* F_σ *P-ideal. Then there exists a Laver large set contained in one equivalence class of* E*.*

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Let $E \subseteq \omega^\omega \times \omega^\omega$ be an equivalence relation containing K, i.e. $E \supseteq K$, which is Borel reducible to $E_{\mathcal{I}}$ for some F_σ P-ideal. Then there exists a Laver large set contained in one equivalence class of E.

Proof. Consider the set
$X = \{x \in \omega^\omega : [x]_E \text{ contains all branches of some Laver tree}\}$.

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Proof. Consider the set
$X = \{x \in \omega^\omega : [x]_E \text{ contains all branches of some Laver tree}\}$.

- It is non-empty: Otherwise, by the previous theorem there is a Laver tree S such that $E \restriction [S] = \mathrm{id}([S])$. There are $x, y \in [S]$ such that xKy and since $E \supseteq K$, also xEy, a contradiction.

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- It is E-equivalent: For, if $[x]_E$ contained all branches of T_x and $[y]_E$ all branches of T_y, then there would be $z_0 \in T_x$ and $z_1 \in T_y$ such that $z_0 K z_1$, thus $z_0 E z_1$, a contradiction.

Corollaries

- It is a single class and it is Laver large (the complement is in the ideal): Otherwise, the complement would contain all branches of some Laver tree S and again, there would have to be $x \in X$ and $y \in [S]$ such that xEy.

□

Silver dichotomy

Combining with the results from the book of Kanovei, Sabok, Zapletal, the following Silver type dichotomy holds (under the assumption $\forall x \in \mathbb{R}(\omega_1^{L[x]} < \omega_1)$):

Theorem (Silver dichotomy)

Let $E \subseteq \omega^\omega \times \omega^\omega$ be an equivalence relation Borel reducible to $E_{\mathcal{I}}$ for some F_σ P-ideal on ω. Then either $\omega^\omega = (\bigcup_{n\in\omega} E_n) \cup J$, where E_n for every n is an equivalence class of E and J is a set in the Laver ideal, or there exists a Laver tree T such that $E \restriction [T] = \mathrm{id}([T])$.

Silver dichotomy

Corollary

Let $E \subseteq \omega^\omega \times \omega^\omega$ be an equivalence relation Borel reducible to $E_{\mathcal{I}}$ for some F_σ P-ideal and let $X \subseteq \omega^\omega$ be an arbitrary subset (not necessarily definable) such that $\forall x, y \in X (x \not E y)$. Then there exists a Laver tree T such that $E \upharpoonright [T] = \mathrm{id}([T])$.

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Let $E \subseteq \omega^\omega \times \omega^\omega$ be an equivalence relation Borel reducible to $E_{\mathcal{I}}$ for some F_σ P-ideal and let $X \subseteq \omega^\omega$ be an arbitrary subset (not necessarily definable) such that $\forall x, y \in X (x \not E y)$. Then there exists a Laver tree T such that $E \restriction [T] = \mathrm{id}([T])$.

Proof.

The first possibility of the Silver dichotomy cannot happen. If $\omega^\omega = (\bigcup_{n\in\omega} E_n) \cup J$ as in the statement of the previous theorem, then $X \setminus J$ is still not in the Laver ideal and is uncountable. □