Embeddings of spaces of the form $C(K)$

Mirna Džamonja

University of East Anglia, UK

10 July, 2012

Isometries and isomorphisms

Let X and Y be Banach spaces.

Isometries and isomorphisms

Let X and Y be Banach spaces.

Definition

A linear mapping $T : X \to Y$ is an *isometry* if for all $x, y \in X$,

Isometries and isomorphisms

Let X and Y be Banach spaces.

Definition

A linear mapping $T : X \to Y$ is an *isometry* if for all $x, y \in X$,

$$||x - y|| = ||Tx - Ty||.$$

Isometries and isomorphisms

Let X and Y be Banach spaces.

Definition

A linear mapping $T : X \to Y$ is an *isometry* if for all $x, y \in X$,

$$||x - y|| = ||Tx - Ty||.$$

It is an *isomorphism* if there is a constant $c > 0$ such that for all $x \in X$,

Isometries and isomorphisms

Let X and Y be Banach spaces.

Definition

A linear mapping $T : X \to Y$ is an *isometry* if for all $x, y \in X$,

$$||x - y|| = ||Tx - Ty||.$$

It is an *isomorphism* if there is a constant $c > 0$ such that for all $x \in X$,

$$1/c||x|| \leq ||Tx|| \leq c||x||.$$

Isometries and isomorphisms

Let X and Y be Banach spaces.

Definition

A linear mapping $T : X \to Y$ is an *isometry* if for all $x, y \in X$,

$$||x - y|| = ||Tx - Ty||.$$

It is an *isomorphism* if there is a constant $c > 0$ such that for all $x \in X$,

$$1/c||x|| \leq ||Tx|| \leq c||x||.$$

Isometries preserve basically all properties of a space,

Isometries and isomorphisms

Let X and Y be Banach spaces.

Definition

A linear mapping $T : X \to Y$ is an *isometry* if for all $x, y \in X$,

$$||x - y|| = ||Tx - Ty||.$$

It is an *isomorphism* if there is a constant $c > 0$ such that for all $x \in X$,

$$1/c||x|| \leq ||Tx|| \leq c||x||.$$

Isometries preserve basically all properties of a space, it is not always clear what is preserved by isomorphism.

The universality number

Definition

Given a quasi-order $(\mathcal{K}, \leq)$,

The universality number

Definition

Given a quasi-order $(\mathcal{K}, \leq)$, *the universality number* of $(\mathcal{K}, \leq)$ ($\mathrm{univ}(\mathcal{K}, \leq)$) is the smallest size of a family $\mathcal{L} \subseteq \mathcal{K}$ such that

The universality number

Definition

Given a quasi-order $(\mathcal{K}, \leq)$, *the universality number* of $(\mathcal{K}, \leq)$ ($\mathrm{univ}(\mathcal{K}, \leq)$) is the smallest size of a family $\mathcal{L} \subseteq \mathcal{K}$ such that

$$(\forall K \in \mathcal{K})(\exists L \in \mathcal{L})(K \leq L).$$

The universality number

Definition

Given a quasi-order $(\mathcal{K}, \leq)$, *the universality number* of $(\mathcal{K}, \leq)$ ($\mathrm{univ}(\mathcal{K}, \leq)$) is the smallest size of a family $\mathcal{L} \subseteq \mathcal{K}$ such that

$$(\forall K \in \mathcal{K})(\exists L \in \mathcal{L})(K \leq L).$$

We shall be interested in the universality number for the classes of the form:

The universality number

Definition

Given a quasi-order $(\mathcal{K}, \leq)$, *the universality number* of $(\mathcal{K}, \leq)$ ($\mathrm{univ}(\mathcal{K}, \leq)$) is the smallest size of a family $\mathcal{L} \subseteq \mathcal{K}$ such that

$$(\forall K \in \mathcal{K})(\exists L \in \mathcal{L})(K \leq L).$$

We shall be interested in the universality number for the classes of the form:

- $\mathcal{K}$ is the class of Banach spaces of a given density κ, $\leq$ is an isometric embedding, denoted $(\mathfrak{B}_\kappa, \leq_m)$,

The universality number

Definition

Given a quasi-order $(\mathcal{K}, \leq)$, *the universality number* of $(\mathcal{K}, \leq)$ ($\mathrm{univ}(\mathcal{K}, \leq)$) is the smallest size of a family $\mathcal{L} \subseteq \mathcal{K}$ such that

$$(\forall K \in \mathcal{K})(\exists L \in \mathcal{L})(K \leq L).$$

We shall be interested in the universality number for the classes of the form:

- $\mathcal{K}$ is the class of Banach spaces of a given density κ, $\leq$ is an isometric embedding, denoted $(\mathfrak{B}_\kappa, \leq_m)$,
- $\mathcal{K}$ is as above, but $\leq$ is an isomorphic embedding, denoted $(\mathfrak{B}_\kappa, \leq_i)$.

The universality number

Definition

Given a quasi-order $(\mathcal{K}, \leq)$, *the universality number* of $(\mathcal{K}, \leq)$ $(\mathrm{univ}(\mathcal{K}, \leq))$ is the smallest size of a family $\mathcal{L} \subseteq \mathcal{K}$ such that

$$(\forall K \in \mathcal{K})(\exists L \in \mathcal{L})(K \leq L).$$

We shall be interested in the universality number for the classes of the form:

- $\mathcal{K}$ is the class of Banach spaces of a given density κ, $\leq$ is an isometric embedding, denoted $(\mathfrak{B}_\kappa, \leq_m)$,
- $\mathcal{K}$ is as above, but $\leq$ is an isomorphic embedding, denoted $(\mathfrak{B}_\kappa, \leq_i)$.

These questions, in particular looking for a single universal element in these classes follow both a classical

The universality number

Definition

Given a quasi-order $(\mathcal{K}, \leq)$, *the universality number* of $(\mathcal{K}, \leq)$ ($\mathrm{univ}(\mathcal{K}, \leq)$) is the smallest size of a family $\mathcal{L} \subseteq \mathcal{K}$ such that

$$(\forall K \in \mathcal{K})(\exists L \in \mathcal{L})(K \leq L).$$

We shall be interested in the universality number for the classes of the form:

- $\mathcal{K}$ is the class of Banach spaces of a given density κ, $\leq$ is an isometric embedding, denoted $(\mathfrak{B}_\kappa, \leq_m)$,
- $\mathcal{K}$ is as above, but $\leq$ is an isomorphic embedding, denoted $(\mathfrak{B}_\kappa, \leq_i)$.

These questions, in particular looking for a single universal element in these classes follow both a classical (e.g. Banach, Bourgain, Szlenk)

The universality number

Definition

Given a quasi-order $(\mathcal{K}, \leq)$, *the universality number* of $(\mathcal{K}, \leq)$ $(\mathrm{univ}(\mathcal{K}, \leq))$ is the smallest size of a family $\mathcal{L} \subseteq \mathcal{K}$ such that

$$(\forall K \in \mathcal{K})(\exists L \in \mathcal{L})(K \leq L).$$

We shall be interested in the universality number for the classes of the form:

- $\mathcal{K}$ is the class of Banach spaces of a given density κ, $\leq$ is an isometric embedding, denoted $(\mathfrak{B}_\kappa, \leq_m)$,
- $\mathcal{K}$ is as above, but $\leq$ is an isomorphic embedding, denoted $(\mathfrak{B}_\kappa, \leq_i)$.

These questions, in particular looking for a single universal element in these classes follow both a classical (e.g. Banach, Bourgain, Szlenk) and modern tradition

The universality number

Definition

Given a quasi-order $(\mathcal{K}, \leq)$, *the universality number* of $(\mathcal{K}, \leq)$ $(\mathrm{univ}(\mathcal{K}, \leq))$ is the smallest size of a family $\mathcal{L} \subseteq \mathcal{K}$ such that

$$(\forall K \in \mathcal{K})(\exists L \in \mathcal{L})(K \leq L).$$

We shall be interested in the universality number for the classes of the form:

- $\mathcal{K}$ is the class of Banach spaces of a given density κ, $\leq$ is an isometric embedding, denoted $(\mathfrak{B}_\kappa, \leq_m)$,
- $\mathcal{K}$ is as above, but $\leq$ is an isomorphic embedding, denoted $(\mathfrak{B}_\kappa, \leq_i)$.

These questions, in particular looking for a single universal element in these classes follow both a classical (e.g. Banach, Bourgain, Szlenk) and modern tradition (e.g. Brech-Koszmider, Dž.-Soukup, Krupski-Marciszewski, Shelah-Usvyatsov, Todorčević).

The universality number

Definition

Given a quasi-order $(\mathcal{K}, \leq)$, *the universality number* of $(\mathcal{K}, \leq)$ $(\mathrm{univ}(\mathcal{K}, \leq))$ is the smallest size of a family $\mathcal{L} \subseteq \mathcal{K}$ such that

$$(\forall K \in \mathcal{K})(\exists L \in \mathcal{L})(K \leq L).$$

We shall be interested in the universality number for the classes of the form:

- $\mathcal{K}$ is the class of Banach spaces of a given density κ, $\leq$ is an isometric embedding, denoted $(\mathfrak{B}_\kappa, \leq_m)$,
- $\mathcal{K}$ is as above, but $\leq$ is an isomorphic embedding, denoted $(\mathfrak{B}_\kappa, \leq_i)$.

These questions, in particular looking for a single universal element in these classes follow both a classical (e.g. Banach, Bourgain, Szlenk) and modern tradition (e.g. Brech-Koszmider, Dž.-Soukup, Krupski-Marciszewski, Shelah-Usvyatsov, Todorčević). Some results will be mentioned.

Some simplificatons

If $\mathfrak{A}$ is a Boolean algebra, by $\mathrm{St}(\mathfrak{A})$ we denote its Stone space.

Some simplificatons

If $\mathfrak{A}$ is a Boolean algebra, by $\mathrm{St}(\mathfrak{A})$ we denote its Stone space. The size of $\mathfrak{A}$ is the density of $C(\mathrm{St}(\mathfrak{A}))$.

Some simplificatons

If $\mathfrak{A}$ is a Boolean algebra, by $\mathrm{St}(\mathfrak{A})$ we denote its Stone space. The size of $\mathfrak{A}$ is the density of $C(\mathrm{St}(\mathfrak{A}))$. Here are some simplifications and observations.

Some simplificatons

If $\mathfrak{A}$ is a Boolean algebra, by $\mathrm{St}(\mathfrak{A})$ we denote its Stone space. The size of $\mathfrak{A}$ is the density of $C(\mathrm{St}(\mathfrak{A}))$. Here are some simplifications and observations.

Theorem

(Brech-Koszmider, folklore)
(1) Every Banach space X of density κ isometrically embeds into one of the form $C(\mathrm{St}(\mathfrak{A}))$, of the same density.

Some simplificatons

If $\mathfrak{A}$ is a Boolean algebra, by $\mathrm{St}(\mathfrak{A})$ we denote its Stone space. The size of $\mathfrak{A}$ is the density of $C(\mathrm{St}(\mathfrak{A}))$. Here are some simplifications and observations.

Theorem

(Brech-Koszmider, folklore)
(1) Every Banach space X of density κ isometrically embeds into one of the form $C(\mathrm{St}(\mathfrak{A}))$, of the same density.
(2) If $\mathfrak{A}$ embeds into $\mathfrak{B}$ then $C(\mathrm{St}(\mathfrak{A}))$ isometrically embeds into $C(\mathrm{St}(\mathfrak{B}))$.

Some simplificatons

If $\mathfrak{A}$ is a Boolean algebra, by $\mathrm{St}(\mathfrak{A})$ we denote its Stone space. The size of $\mathfrak{A}$ is the density of $C(\mathrm{St}(\mathfrak{A}))$. Here are some simplifications and observations.

Theorem

(Brech-Koszmider, folklore)
(1) Every Banach space X of density κ isometrically embeds into one of the form $C(\mathrm{St}(\mathfrak{A}))$, of the same density.
(2) If $\mathfrak{A}$ embeds into $\mathfrak{B}$ then $C(\mathrm{St}(\mathfrak{A}))$ isometrically embeds into $C(\mathrm{St}(\mathfrak{B}))$.

Let $(\mathcal{A}_\kappa, \leq_e)$ denote the class of Boolean algebras of size κ with the embeddability relation.

Some simplificatons

If $\mathfrak{A}$ is a Boolean algebra, by $\mathrm{St}(\mathfrak{A})$ we denote its Stone space. The size of $\mathfrak{A}$ is the density of $C(\mathrm{St}(\mathfrak{A}))$. Here are some simplifications and observations.

Theorem

(Brech-Koszmider, folklore)
(1) Every Banach space X of density κ isometrically embeds into one of the form $C(\mathrm{St}(\mathfrak{A}))$, of the same density.
(2) If $\mathfrak{A}$ embeds into $\mathfrak{B}$ then $C(\mathrm{St}(\mathfrak{A}))$ isometrically embeds into $C(\mathrm{St}(\mathfrak{B}))$.

Let $(\mathcal{A}_\kappa, \leq_e)$ denote the class of Boolean algebras of size κ with the embeddability relation.

Corollary

$$\mathrm{univ}((\mathfrak{B}_\kappa, \leq_i)) \leq \mathrm{univ}((\mathfrak{B}_\kappa, \leq_m)) \leq \mathrm{univ}((\mathcal{A}_\kappa, \leq_e)).$$

Some of the known theorems

(1) (Banach 1930) $C[0,1]$ is a universal separable Banach space.

Some of the known theorems

(1) (Banach 1930) $C[0,1]$ is a universal separable Banach space.
(2) (classical model theory) Under GCH there is a universal Boolean algebra at every cardinal - so also an isometrically universal Banach space.

Some of the known theorems

(1) (Banach 1930) $C[0,1]$ is a universal separable Banach space.
(2) (classical model theory) Under GCH there is a universal Boolean algebra at every cardinal - so also an isometrically universal Banach space.
(3) (Parovichenko 1951) Under CH, $\mathcal{P}(\omega)/fin$ is a universal Boolean algebra –so l_∞/c_0 is a universal Banach space of density $\aleph_1$.

Some of the known theorems

(1) (Banach 1930) $C[0,1]$ is a universal separable Banach space.
(2) (classical model theory) Under GCH there is a universal Boolean algebra at every cardinal - so also an isometrically universal Banach space.
(3) (Parovichenko 1951) Under CH, $\mathcal{P}(\omega)/fin$ is a universal Boolean algebra –so l_∞/c_0 is a universal Banach space of density $\aleph_1$.
(4) (Brech-Koszmider 2011) In the Cohen model for $\aleph_2$ Cohen reals over a model of GCH, there is an isometrically universal Banach space of density $\omega_2 = \mathfrak{c}$ but l_∞/c_0 is not even isomorphically universal.

Some of the known theorems

(1) (Banach 1930) $C[0,1]$ is a universal separable Banach space.
(2) (classical model theory) Under GCH there is a universal Boolean algebra at every cardinal - so also an isometrically universal Banach space.
(3) (Parovichenko 1951) Under CH, $\mathcal{P}(\omega)/fin$ is a universal Boolean algebra –so l_∞/c_0 is a universal Banach space of density $\aleph_1$.
(4) (Brech-Koszmider 2011) In the Cohen model for $\aleph_2$ Cohen reals over a model of GCH, there is an isometrically universal Banach space of density $\omega_2 = \mathfrak{c}$ but l_∞/c_0 is not even isomorphically universal.
(5) (Džamonja-Soukup, preprint 2010, after the BK preprint) The same as (4) but with any number $\geq \aleph_2$ of Cohen reals.

Some of the known theorems

(1) (Banach 1930) $C[0,1]$ is a universal separable Banach space.
(2) (classical model theory) Under GCH there is a universal Boolean algebra at every cardinal - so also an isometrically universal Banach space.
(3) (Parovichenko 1951) Under CH, $\mathcal{P}(\omega)/fin$ is a universal Boolean algebra –so l_∞/c_0 is a universal Banach space of density $\aleph_1$.
(4) (Brech-Koszmider 2011) In the Cohen model for $\aleph_2$ Cohen reals over a model of GCH, there is an isometrically universal Banach space of density $\omega_2 = \mathfrak{c}$ but l_∞/c_0 is not even isomorphically universal.
(5) (Džamonja-Soukup, preprint 2010, after the BK preprint) The same as (4) but with any number $\geq \aleph_2$ of Cohen reals.

(6) (Kojman-Shelah 1994) $\mathrm{univ}((\mathcal{A}_\kappa, \leq_e))$ is the same as the universality number of linear orders under embeddings, which is large as soon as GCH fails sufficiently.

(6) (Kojman-Shelah 1994) $\mathrm{univ}((\mathcal{A}_\kappa, \leq_e))$ is the same as the universality number of linear orders under embeddings, which is large as soon as GCH fails sufficiently.
(7) (Shelah-Usvyatsov 2006) The same as (6) but for $\mathrm{univ}(\mathfrak{B}_\kappa, \leq_m)$.

(6) (Kojman-Shelah 1994) $\mathrm{univ}((\mathcal{A}_\kappa, \leq_e))$ is the same as the universality number of linear orders under embeddings, which is large as soon as GCH fails sufficiently.
(7) (Shelah-Usvyatsov 2006) The same as (6) but for $\mathrm{univ}(\mathfrak{B}_\kappa, \leq_m)$.

In none of the known models is any of the inequalities $\mathrm{univ}((\mathfrak{B}_\kappa, \leq_i)) \leq \mathrm{univ}((\mathfrak{B}_\kappa, \leq_m)) \leq \mathrm{univ}((\mathcal{A}_\kappa, \leq_e))$ known to be strict, for any κ.

(6) (Kojman-Shelah 1994) $\mathrm{univ}((\mathcal{A}_\kappa, \leq_e))$ is the same as the universality number of linear orders under embeddings, which is large as soon as GCH fails sufficiently.
(7) (Shelah-Usvyatsov 2006) The same as (6) but for $\mathrm{univ}(\mathfrak{B}_\kappa, \leq_m)$.

In none of the known models is any of the inequalities $\mathrm{univ}((\mathfrak{B}_\kappa, \leq_i)) \leq \mathrm{univ}((\mathfrak{B}_\kappa, \leq_m)) \leq \mathrm{univ}((\mathcal{A}_\kappa, \leq_e))$ known to be strict, for any κ.

Conjecture
$\mathrm{univ}((\mathfrak{B}_\kappa, \leq_i)) = \mathrm{univ}((\mathfrak{B}_\kappa, \leq_m)) = \mathrm{univ}((\mathcal{A}_\kappa, \leq_e))$.

Some new results

A natural place to start is density $\aleph_1$ and something like the Cohen model.

Some new results

A natural place to start is density $\aleph_1$ and something like the Cohen model.

Fix a set $A = \{a_\alpha : \alpha < \omega_1\}$ of indices.

Definition

The forcing $\mathbb{P}$ consists of Boolean algebras p generated by some finite subset w_p of A satisfying $p \cap A = w_p$.

Some new results

A natural place to start is density $\aleph_1$ and something like the Cohen model.

Fix a set $A = \{a_\alpha : \alpha < \omega_1\}$ of indices.

Definition

The forcing $\mathbb{P}$ consists of Boolean algebras p generated by some finite subset w_p of A satisfying $p \cap A = w_p$. The ordering on $\mathbb{P}$ is given by $p \leq q$ if p is embeddable as a subalgebra of q and the embedding fixes w_p.

Some new results

A natural place to start is density $\aleph_1$ and something like the Cohen model.

Fix a set $A = \{a_\alpha : \alpha < \omega_1\}$ of indices.

Definition

The forcing $\mathbb{P}$ consists of Boolean algebras p generated by some finite subset w_p of A satisfying $p \cap A = w_p$. The ordering on $\mathbb{P}$ is given by $p \leq q$ if p is embeddable as a subalgebra of q and the embedding fixes w_p.

Lemma

(1) $\mathbb{P}$ adds a Boolean algebra generated by $\{a_\alpha : \alpha < \omega_1\}$ and
(2) $\mathbb{P}$ satisfies the ccc, in fact it has the property of Knaster (and more).

Some new results

A natural place to start is density $\aleph_1$ and something like the Cohen model.

Fix a set $A = \{a_\alpha : \alpha < \omega_1\}$ of indices.

Definition

The forcing $\mathbb{P}$ consists of Boolean algebras p generated by some finite subset w_p of A satisfying $p \cap A = w_p$. The ordering on $\mathbb{P}$ is given by $p \leq q$ if p is embeddable as a subalgebra of q and the embedding fixes w_p.

Lemma

(1) $\mathbb{P}$ adds a Boolean algebra generated by $\{a_\alpha : \alpha < \omega_1\}$ and

(2) $\mathbb{P}$ satisfies the ccc, in fact it has the property of Knaster (and more).

Theorem

If $\mathfrak{A}$ is the generic Boolean algebra, then there is no Banach space X in the ground model such that $C(\mathrm{St}(\mathfrak{A}))$ isomorphically embeds into X in the extension by $\mathbb{P}$.

Proof.

- Suppose X is such, T is the embedding. Let p^* force that, WLOG p^* decides $c > 0$ such that $1/c||x|| \leq ||\dot{T}x|| \leq c||x||$.

Theorem

If $\mathfrak{A}$ is the generic Boolean algebra, then there is no Banach space X in the ground model such that $C(\mathrm{St}(\mathfrak{A}))$ isomorphically embeds into X in the extension by $\mathbb{P}$.

Proof.

- Suppose X is such, T is the embedding. Let p^* force that, WLOG p^* decides $c > 0$ such that $1/c||x|| \leq ||Tx|| \leq c||x||$.
- Let n_* be such that $c < n_*$.

Theorem

If $\mathfrak{A}$ is the generic Boolean algebra, then there is no Banach space X in the ground model such that $C(\mathrm{St}(\mathfrak{A}))$ isomorphically embeds into X in the extension by $\mathbb{P}$.

Proof.

- Suppose X is such, T is the embedding. Let p^* force that, WLOG p^* decides $c > 0$ such that $1/c||x|| \leq ||Tx|| \leq c||x||$.
- Let n_* be such that $c < n_*$.
- For $i < \omega_1$ let $p_i \geq p^*$ such that $a_i \in w_{p_i}$ and WLOG p_i decides the value $x_i = T(\chi_{[a_i]})$.

Theorem

If $\mathfrak{A}$ is the generic Boolean algebra, then there is no Banach space X in the ground model such that $C(\mathrm{St}(\mathfrak{A}))$ isomorphically embeds into X in the extension by $\mathbb{P}$.

Proof.

- Suppose X is such, T is the embedding. Let p^* force that, WLOG p^* decides $c > 0$ such that $1/c||x|| \leq ||Tx|| \leq c||x||$.
- Let n_* be such that $c < n_*$.
- For $i < \omega_1$ let $p_i \geq p^*$ such that $a_i \in w_{p_i}$ and WLOG p_i decides the value $x_i = T(\chi_{[a_i]})$.
- Assume WLOG Δ-system $w^*, w_i, \ldots$. Since $a_i \in w_{p_i}$, we can assume that $w_i \neq w^*$, and in particular that $a_i \in w_i \setminus w^*$.

Theorem

If $\mathfrak{A}$ is the generic Boolean algebra, then there is no Banach space X in the ground model such that $C(\mathrm{St}(\mathfrak{A}))$ isomorphically embeds into X in the extension by $\mathbb{P}$.

Proof.

- Suppose X is such, T is the embedding. Let p^* force that, WLOG p^* decides $c > 0$ such that $1/c||x|| \leq ||Tx|| \leq c||x||$.
- Let n_* be such that $c < n_*$.
- For $i < \omega_1$ let $p_i \geq p^*$ such that $a_i \in w_{p_i}$ and WLOG p_i decides the value $x_i = T(\chi_{[a_i]})$.
- Assume WLOG Δ-system $w^*, w_i, \ldots$. Since $a_i \in w_{p_i}$, we can assume that $w_i \neq w^*$, and in particular that $a_i \in w_i \setminus w^*$.

Theorem

If $\mathfrak{A}$ is the generic Boolean algebra, then there is no Banach space X in the ground model such that $C(\mathrm{St}(\mathfrak{A}))$ isomorphically embeds into X in the extension by $\mathbb{P}$.

Proof.

- Suppose X is such, T is the embedding. Let p^* force that, WLOG p^* decides $c > 0$ such that $1/c||x|| \le ||Tx|| \le c||x||$.
- Let n_* be such that $c < n_*$.
- For $i < \omega_1$ let $p_i \ge p^*$ such that $a_i \in w_{p_i}$ and WLOG p_i decides the value $x_i = T(\chi_{[a_i]})$.
- Assume WLOG Δ-system $w^*, w_i, \ldots$. Since $a_i \in w_{p_i}$, we can assume that $w_i \ne w^*$, and in particular that $a_i \in w_i \setminus w^*$. We also assume that for all $i < j$ there is isomorphism $h_{i,j} : w_i \to w_j$ over w^*.

□

- Let $u_{i,j}$ be the set of equations obtained by identifying each x with $h_{i,j}(x)$.

- Let $u_{i,j}$ be the set of equations obtained by identifying each x with $h_{i,j}(x)$.
- Let us take $i_0 \neq i_1 \neq \ldots i_{n_*^2}$.

- Let $u_{i,j}$ be the set of equations obtained by identifying each x with $h_{i,j}(x)$.
- Let us take $i_0 \neq i_1 \neq \ldots i_{n_*^2}$.

- Let $u_{i,j}$ be the set of equations obtained by identifying each x with $h_{i,j}(x)$.
- Let us take $i_0 \neq i_1 \neq \dots i_{n_*^2}$. Since there are no equations in $w_i \cup w_j \cup u_{i,j}$ that connect a_i and a_j for $i \neq j$, we can find two conditions q' and q'' which both extend $p_{i_0}, \dots p_{n_*^2}$ and such that in q' we have $a_{i_0} \leq \dots a_{i_{n_*^2}}$ and in q'' we have that $a_{i_0}, \dots a_{i_{n_*^2}}$ are pairwise disjoint.

- Let $u_{i,j}$ be the set of equations obtained by identifying each x with $h_{i,j}(x)$.
- Let us take $i_0 \neq i_1 \neq \dots i_{n_*^2}$. Since there are no equations in $w_i \cup w_j \cup u_{i,j}$ that connect a_i and a_j for $i \neq j$, we can find two conditions q' and q'' which both extend $p_{i_0}, \dots p_{n_*^2}$ and such that in q' we have $a_{i_0} \leq \dots a_{i_{n_*^2}}$ and in q'' we have that $a_{i_0}, \dots a_{i_{n_*^2}}$ are pairwise disjoint. Hence q' forces that $\chi_{[a_{i_0}]} + \dots \chi_{[a_{i_{n_*^2}}]}$ has norm $n_*^2 + 1$ and q'' forces that it has norm 1.

- Let $u_{i,j}$ be the set of equations obtained by identifying each x with $h_{i,j}(x)$.
- Let us take $i_0 \neq i_1 \neq \ldots i_{n_*^2}$. Since there are no equations in $w_i \cup w_j \cup u_{i,j}$ that connect a_i and a_j for $i \neq j$, we can find two conditions q' and q'' which both extend $p_{i_0}, \ldots p_{n_*^2}$ and such that in q' we have $a_{i_0} \leq \ldots a_{i_{n_*^2}}$ and in q'' we have that $a_{i_0}, \ldots a_{i_{n_*^2}}$ are pairwise disjoint. Hence q' forces that $\chi_{[a_{i_0}]} + \ldots \chi_{[a_{i_{n_*^2}}]}$ has norm $n_*^2 + 1$ and q'' forces that it has norm 1. So the vector $x_{i_0} + \ldots x_{i_{n_*^2}}$ must have norm both $< n_*$ and $> n_*$ in X, which is impossible.

QED

- Let $u_{i,j}$ be the set of equations obtained by identifying each x with $h_{i,j}(x)$.
- Let us take $i_0 \neq i_1 \neq \ldots i_{n_*^2}$. Since there are no equations in $w_i \cup w_j \cup u_{i,j}$ that connect a_i and a_j for $i \neq j$, we can find two conditions q' and q'' which both extend $p_{i_0}, \ldots p_{n_*^2}$ and such that in q' we have $a_{i_0} \leq \ldots a_{i_{n_*^2}}$ and in q'' we have that $a_{i_0}, \ldots a_{i_{n_*^2}}$ are pairwise disjoint. Hence q' forces that $\chi_{[a_{i_0}]} + \ldots \chi_{[a_{i_{n_*^2}}]}$ has norm $n_*^2 + 1$ and q'' forces that it has norm 1. So the vector $x_{i_0} + \ldots x_{i_{n_*^2}}$ must have norm both $< n_*$ and $> n_*$ in X, which is impossible.

QED

Now of course we want to iterate in length ω_2, every Boolean algebra of size $\aleph_1$ is in an intermediate model, bla

- Let $u_{i,j}$ be the set of equations obtained by identifying each x with $h_{i,j}(x)$.
- Let us take $i_0 \neq i_1 \neq \ldots i_{n_*^2}$. Since there are no equations in $w_i \cup w_j \cup u_{i,j}$ that connect a_i and a_j for $i \neq j$, we can find two conditions q' and q'' which both extend $p_{i_0}, \ldots p_{n_*^2}$ and such that in q' we have $a_{i_0} \leq \ldots a_{i_{n_*^2}}$ and in q'' we have that $a_{i_0}, \ldots a_{i_{n_*^2}}$ are pairwise disjoint. Hence q' forces that $\chi_{[a_{i_0}]} + \ldots \chi_{[a_{i_{n_*^2}}]}$ has norm $n_*^2 + 1$ and q'' forces that it has norm 1. So the vector $x_{i_0} + \ldots x_{i_{n_*^2}}$ must have norm both $< n_*$ and $> n_*$ in X, which is impossible.

QED

Now of course we want to iterate in length ω_2, every Boolean algebra of size $\aleph_1$ is in an intermediate model, bla *STOP*

- Let $u_{i,j}$ be the set of equations obtained by identifying each x with $h_{i,j}(x)$.
- Let us take $i_0 \neq i_1 \neq \dots i_{n_*^2}$. Since there are no equations in $w_i \cup w_j \cup u_{i,j}$ that connect a_i and a_j for $i \neq j$, we can find two conditions q' and q'' which both extend $p_{i_0}, \dots p_{n_*^2}$ and such that in q' we have $a_{i_0} \leq \dots a_{i_{n_*^2}}$ and in q'' we have that $a_{i_0}, \dots a_{i_{n_*^2}}$ are pairwise disjoint. Hence q' forces that $\chi_{[a_{i_0}]} + \dots \chi_{[a_{i_{n_*^2}}]}$ has norm $n_*^2 + 1$ and q'' forces that it has norm 1. So the vector $x_{i_0} + \dots x_{i_{n_*^2}}$ must have norm both $< n_*$ and $> n_*$ in X, which is impossible.

QED

Now of course we want to iterate in length ω_2, every Boolean algebra of size $\aleph_1$ is in an intermediate model, bla *STOP* even if $\mathfrak{A}$ is in an intermediate universe, we cannot say that $C(\mathrm{St}(\mathfrak{A}))$ is, since new reals keep being added and hence so do new elements of $C(\mathrm{St}(\mathfrak{A}))$.

I know how to overcome this problem in two situations:

I know how to overcome this problem in two situations:

- isometries and

I know how to overcome this problem in two situations:

- isometries and

I know how to overcome this problem in two situations:

- isometries and
- densities larger than the continuum.

I know how to overcome this problem in two situations:

- isometries and
- densities larger than the continuum.

I know how to overcome this problem in two situations:

- isometries and
- densities larger than the continuum.

Theorem

(1) Let G be a generic for the iteration with finite supports of the forcing to add the generic Boolean algebra of size $\aleph_1$ *by finite conditions over a model of GCH.*

I know how to overcome this problem in two situations:

- isometries and
- densities larger than the continuum.

Theorem

(1) Let G be a generic for the iteration with finite supports of the forcing to add the generic Boolean algebra of size $\aleph_1$ *by finite conditions over a model of GCH. Then in* $V[G]$ *the universality number of the class of Banach spaces of density* $\aleph_1$ *under* isometries *is* $\aleph_2$*.*

I know how to overcome this problem in two situations:

- isometries and
- densities larger than the continuum.

Theorem

(1) Let G be a generic for the iteration with finite supports of the forcing to add the generic Boolean algebra of size $\aleph_1$ by finite conditions over a model of GCH.Then in $V[G]$ the universality number of the class of Banach spaces of density $\aleph_1$ under isometries *is $\aleph_2$.*
(2) (modulo checking an iteration theorem)

I know how to overcome this problem in two situations:

- isometries and
- densities larger than the continuum.

Theorem

(1) Let G be a generic for the iteration with finite supports of the forcing to add the generic Boolean algebra of size $\aleph_1$ *by finite conditions over a model of GCH.Then in* $V[G]$ *the universality number of the class of Banach spaces of density* $\aleph_1$ *under* isometries *is* $\aleph_2$.

(2) (modulo checking an iteration theorem) Let $\lambda = \lambda^{<\lambda} > \aleph_0$.

I know how to overcome this problem in two situations:

- isometries and
- densities larger than the continuum.

Theorem

(1) Let G be a generic for the iteration with finite supports of the forcing to add the generic Boolean algebra of size $\aleph_1$ *by finite conditions over a model of GCH.Then in* $V[G]$ *the universality number of the class of Banach spaces of density* $\aleph_1$ *under* isometries *is* $\aleph_2$*.*
(2) (modulo checking an iteration theorem) Let $\lambda = \lambda^{<\lambda} > \aleph_0$*. Let G be a generic for the iteration with* $(< \lambda)$ *supports of the forcing to add the generic Boolean algebra of size* λ^+ *by conditions of size* $< \lambda$ *over a model of GCH.*

I know how to overcome this problem in two situations:

- isometries and
- densities larger than the continuum.

Theorem

(1) Let G be a generic for the iteration with finite supports of the forcing to add the generic Boolean algebra of size $\aleph_1$ *by finite conditions over a model of GCH.Then in* $V[G]$ *the universality number of the class of Banach spaces of density* $\aleph_1$ *under* isometries *is* $\aleph_2$.

(2) (modulo checking an iteration theorem) Let $\lambda = \lambda^{<\lambda} > \aleph_0$. *Let G be a generic for the iteration with* $(< \lambda)$ *supports of the forcing to add the generic Boolean algebra of size* λ^+ *by conditions of size* $< \lambda$ *over a model of GCH. Then in* $V[G]$ *the universality number of the class of Banach spaces of density* λ^+ *under* isomorphisms *is* λ^{++}.

Proof.

For (2), use the scenario attempted above, since no new reals are added. Iterability .. .

Proof.

For (2), use the scenario attempted above, since no new reals are added. Iterability .. .
(1) For simplicity, prove that these is no universal.

Proof.

For (2), use the scenario attempted above, since no new reals are added. Iterability .. .

(1) For simplicity, prove that these is no universal. Suppose it is $C(\mathrm{St}(\mathfrak{A}))$,

Proof.

For (2), use the scenario attempted above, since no new reals are added. Iterability .. .

(1) For simplicity, prove that these is no universal. Suppose it is $C(\mathrm{St}(\mathfrak{A}))$, say in the ground model. Let p^* force $\dot{T}$ to be an isometric embedding from the first generic algebra to $C(\mathrm{St}(\mathfrak{A}))$.

Proof.

For (2), use the scenario attempted above, since no new reals are added. Iterability .. .

(1) For simplicity, prove that these is no universal. Suppose it is $C(\mathrm{St}(\mathfrak{A}))$, say in the ground model. Let p^* force $\dot{T}$ to be an isometric embedding from the first generic algebra to $C(\mathrm{St}(\mathfrak{A}))$. Let $p_i \geq p^*$ force that h_i is a simple rational function with rational coefficient with $||\dot{T}(\chi_{[a_i]}) - h_i|| < 1/2$

Proof.

For (2), use the scenario attempted above, since no new reals are added. Iterability .. .

(1) For simplicity, prove that these is no universal. Suppose it is $C(\mathrm{St}(\mathfrak{A}))$, say in the ground model. Let p^* force $\dot{T}$ to be an isometric embedding from the first generic algebra to $C(\mathrm{St}(\mathfrak{A}))$. Let $p_i \geq p^*$ force that h_i is a simple rational function with rational coefficient with $||\dot{T}(\chi_{[a_i]}) - h_i|| < 1/2$ (so $h_i \in V$).

Proof.

For (2), use the scenario attempted above, since no new reals are added. Iterability .. .

(1) For simplicity, prove that these is no universal. Suppose it is $C(\mathrm{St}(\mathfrak{A}))$, say in the ground model. Let p^* force $\dot{T}$ to be an isometric embedding from the first generic algebra to $C(\mathrm{St}(\mathfrak{A}))$. Let $p_i \geq p^*$ force that h_i is a simple rational function with rational coefficient with $||\dot{T}(\chi_{[a_i]}) - h_i|| < 1/2$ (so $h_i \in V$). Do several Δ-system arguments,

Proof.

For (2), use the scenario attempted above, since no new reals are added. Iterability .. .

(1) For simplicity, prove that these is no universal. Suppose it is $C(\mathrm{St}(\mathfrak{A}))$, say in the ground model. Let p^* force $\dot{T}$ to be an isometric embedding from the first generic algebra to $C(\mathrm{St}(\mathfrak{A}))$. Let $p_i \geq p^*$ force that h_i is a simple rational function with rational coefficient with $||\dot{T}(\chi_{[a_i]}) - h_i|| < 1/2$ (so $h_i \in V$). Do several Δ-system arguments, take "clean" $i \neq j$.

Proof.

For (2), use the scenario attempted above, since no new reals are added. Iterability .. .

(1) For simplicity, prove that these is no universal. Suppose it is $C(\mathrm{St}(\mathfrak{A}))$, say in the ground model. Let p^* force $\dot{T}$ to be an isometric embedding from the first generic algebra to $C(\mathrm{St}(\mathfrak{A}))$. Let $p_i \geq p^*$ force that h_i is a simple rational function with rational coefficient with $||\dot{T}(\chi_{[a_i]}) - h_i|| < 1/2$ (so $h_i \in V$). Do several Δ-system arguments, take "clean" $i \neq j$. If $||h_i - h_j|| < 1$, show that there cannot be an extension of $p_i \cup p_j$ forcing a_i and a_j to be disjoint,

Proof.

For (2), use the scenario attempted above, since no new reals are added. Iterability .. .
(1) For simplicity, prove that these is no universal. Suppose it is $C(\mathrm{St}(\mathfrak{A}))$, say in the ground model. Let p^* force $\dot{T}$ to be an isometric embedding from the first generic algebra to $C(\mathrm{St}(\mathfrak{A}))$. Let $p_i \geq p^*$ force that h_i is a simple rational function with rational coefficient with $||\dot{T}(\chi_{[a_i]}) - h_i|| < 1/2$ (so $h_i \in V$). Do several Δ-system arguments, take "clean" $i \neq j$. If $||h_i - h_j|| < 1$, show that there cannot be an extension of $p_i \cup p_j$ forcing a_i and a_j to be disjoint, contradiction. Similarly with $||h_i - h_j|| \geq 1$. □

It is reasonable to assume that:

- (1) above can be improved to isomorphisms, with a better calculation.

It is reasonable to assume that:

- (1) above can be improved to isomorphisms, with a better calculation.

It is reasonable to assume that:

- (1) above can be improved to isomorphisms, with a better calculation. S. Shelah told me he can do it.

It is reasonable to assume that:

- (1) above can be improved to isomorphisms, with a better calculation. S. Shelah told me he can do it.
- The iteration problem, even if the generalized Martin's axiom does not apply directly, should not be serious because we are basically working in the Cohen models.

It is reasonable to assume that:

- (1) above can be improved to isomorphisms, with a better calculation. S. Shelah told me he can do it.
- The iteration problem, even if the generalized Martin's axiom does not apply directly, should not be serious because we are basically working in the Cohen models.

It is reasonable to assume that:

- (1) above can be improved to isomorphisms, with a better calculation. S. Shelah told me he can do it.
- The iteration problem, even if the generalized Martin's axiom does not apply directly, should not be serious because we are basically working in the Cohen models.

The morale is that the conjectures seems to hold as far as the Cohen models are concerned.

It is reasonable to assume that:

- (1) above can be improved to isomorphisms, with a better calculation. S. Shelah told me he can do it.
- The iteration problem, even if the generalized Martin's axiom does not apply directly, should not be serious because we are basically working in the Cohen models.

The morale is that the conjectures seems to hold as far as the Cohen models are concerned.
The next step is to verify the methods of invariants a'la Kojman-Shelah.

We would like a simple model theoretic class $(\mathcal{K}, \leq)$ such that $\mathrm{univ}((\mathcal{K}_\kappa, \leq)) \leq \mathrm{univ}((\mathfrak{B}_\kappa, \leq_i))$ for all relevant κ.

We would like a simple model theoretic class $(\mathcal{K}, \leq)$ such that $\mathrm{univ}((\mathcal{K}_\kappa, \leq)) \leq \mathrm{univ}((\mathfrak{B}_\kappa, \leq_i))$ for all relevant κ. I'll present a candidate.

We work with vector spaces with rational coefficients and with two distinguished unary predicates C, C_0 satisfying $C_0 \subseteq C$.

We work with vector spaces with rational coefficients and with two distinguished unary predicates C, C_0 satisfying $C_0 \subseteq C$. If such a space (V, C, C_0) is the space of sequences of simple rational functions over a Stone space $K = \mathrm{St}(\mathfrak{A})$ and

We work with vector spaces with rational coefficients and with two distinguished unary predicates C, C_0 satisfying $C_0 \subseteq C$. If such a space (V, C, C_0) is the space of sequences of simple rational functions over a Stone space $K = \mathrm{St}(\mathfrak{A})$ and C, C_0 correspond respectively to the set of such sequences which converge or converge to 0, We work with vector spaces with rational coefficients and with two distinguished unary predicates C, C_0 satisfying $C_0 \subseteq C$. If such a space (V, C, C_0) is the space of sequences of simple rational functions over a Stone space $K = \mathrm{St}(\mathfrak{A})$ and C, C_0 correspond respectively to the set of such sequences which converge or converge to 0, then we call (V, C, C_0) a *natural space* and we denote it by $N(\mathfrak{A})$.

Theorem

Suppose that $\mathfrak{A}$ and $\mathfrak{B}$ are Boolean algebras. Then:

Theorem

Suppose that $\mathfrak{A}$ *and* $\mathfrak{B}$ *are Boolean algebras. Then:*
(1) if there is an isomorphic embedding from $C(\mathrm{St}(\mathfrak{A}))$ *to* $C(\mathrm{St}(\mathfrak{B}))$

Theorem

Suppose that $\mathfrak{A}$ and $\mathfrak{B}$ are Boolean algebras. Then:
(1) if there is an isomorphic embedding from $C(\mathrm{St}(\mathfrak{A}))$ to $C(\mathrm{St}(\mathfrak{B}))$ then there is an isomorphic embedding from $N(\mathfrak{A})$ to $N(\mathfrak{B})$, and

Theorem

Suppose that $\mathfrak{A}$ and $\mathfrak{B}$ are Boolean algebras. Then:
(1) if there is an isomorphic embedding from $C(\mathrm{St}(\mathfrak{A}))$ to $C(\mathrm{St}(\mathfrak{B}))$ then there is an isomorphic embedding from $N(\mathfrak{A})$ to $N(\mathfrak{B})$, and
(2) if there is an isomorphic embedding from $N(\mathfrak{A})$ to $N(\mathfrak{B})$ then

Theorem

Suppose that $\mathfrak{A}$ and $\mathfrak{B}$ are Boolean algebras. Then:
(1) if there is an isomorphic embedding from $C(\mathrm{St}(\mathfrak{A}))$ to $C(\mathrm{St}(\mathfrak{B}))$ then there is an isomorphic embedding from $N(\mathfrak{A})$ to $N(\mathfrak{B})$, and
(2) if there is an isomorphic embedding from $N(\mathfrak{A})$ to $N(\mathfrak{B})$ then there is a linear embedding from $C(\mathrm{St}(\mathfrak{A}))$ to $C(\mathrm{St}(\mathfrak{B}))$ which

Theorem

Suppose that $\mathfrak{A}$ and $\mathfrak{B}$ are Boolean algebras. Then:
(1) if there is an isomorphic embedding from $C(\mathrm{St}(\mathfrak{A}))$ to $C(\mathrm{St}(\mathfrak{B}))$ then there is an isomorphic embedding from $N(\mathfrak{A})$ to $N(\mathfrak{B})$, and
(2) if there is an isomorphic embedding from $N(\mathfrak{A})$ to $N(\mathfrak{B})$ then there is a linear embedding from $C(\mathrm{St}(\mathfrak{A}))$ to $C(\mathrm{St}(\mathfrak{B}))$ which maps simple functions with rational coefficients into simple functions with rational coefficients, and preserves the convergence of such sequences.