Set theory and Hausdorff measures

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The following notion is the starting point of geometric measure theory, that is, fractal geometry. The idea is that in the definition of the Lebesgue measure we replace $\inf \sum_i |I_i|$ by $\inf \sum_i |I_i|^d$.

Definition

Let A be a subset of a metric space X. The *d-dimensional Hausdorff measure of A*, denoted by $\mathcal{H}^d(A)$ is defined as follows.

$$\mathcal{H}^d_\delta(A) = \inf \left\{ \sum_{i=1}^{\infty} (\operatorname{diam} U_i)^d : A \subset \bigcup_i U_i, \forall i \operatorname{diam} U_i \leq \delta \right\},$$

$$\mathcal{H}^d(A) = \lim_{\delta \to 0+} \mathcal{H}^d_\delta(A).$$

Remark

For $d = 1, 2, 3$ we get back the classical notions of length, area, volume, but on the one hand this notion is defined for all subsets of a metric space, and on the other hand it makes sense for non-integer d as well.

This allows us to define the next fundamental notion.

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The *Hausdorff dimension of A* is defined as

$$\dim_H A = \inf\{d \geq 0 : \mathcal{H}^d(A) = 0\}.$$

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In fact, for almost all purposes of this talk we will only need the following less technical definition.

Definition

A is of d-dimensional Hausdorff measure zero if for every $\varepsilon > 0$ there is a sequence of balls $B_i(x_i, r_i)$ covering A such that $\sum_i r_i^d < \varepsilon$.

Our first goal is to investigate the σ-ideal of $\mathcal{H}^d$-null sets from the point of view of set theory.
Let us denote this σ-ideal by $\mathcal{N}^d$ (well, if the ambient space is clear).
Let us start with the cardinal invariants.
The next theorem shows their position in the Cichoń Diagram. From now on we will work in $\mathbb{R}^n$.

Theorem (Fremlin)

Let $0 < d < n$. *Then*

- $\mathrm{add}(\mathcal{N}^d) = \mathrm{add}(\mathcal{N})$,
- $\mathrm{cof}(\mathcal{N}^d) = \mathrm{cof}(\mathcal{N})$,
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Question (Fremlin 534Z(a))

Let $0 < d < n$. *Does* $\mathrm{cov}(\mathcal{N}^d) = \mathrm{cov}(\mathcal{N})$ *hold in ZFC?*

Remark

The reason why he asked specifically about this pair is the following. It is not hard to see that $\mathrm{cov}(\mathcal{N}^d) < \mathrm{non}(\mathcal{M})$ and $\mathrm{cov}(\mathcal{M}) < \mathrm{non}(\mathcal{N}^d)$ are consistent with *ZFC*, moreover, we have the following theorem.

Theorem (Shelah-Steprāns)

Let $0 < d < n$. *Then* $\mathrm{non}(\mathcal{N}^d) < \mathrm{non}(\mathcal{N})$ *is consistent with ZFC.*

And here is the answer to Fremlin's question:

Theorem (M.E.-Steprāns)

Let $0 < d < n$. *Then* $\mathrm{cov}(\mathcal{N}^d) > \mathrm{cov}(\mathcal{N})$ *is consistent with ZFC.*

The proof is a rather standard forcing construction building heavily on work of Zapletal.

Question

Let $0 < d_1 < d_2 < n$. *Does* $\mathrm{cov}(\mathcal{N}^{d_1}) = \mathrm{cov}(\mathcal{N}^{d_2})$ *hold in ZFC? (Same for non?)*

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Isomorphism of measures

Question (Weiss-Preiss)

Let $0 < d_1 < d_2 < n$. Are the measures $\mathcal{H}^{d_1}$ and $\mathcal{H}^{d_2}$ isomorphic?

Yes, under *CH*:

Theorem (M.E.)

(CH) Let $0 < d_1 < d_2 < n$. Then the measure spaces $(\mathbb{R}^n, \mathcal{M}^{d_1}, \mathcal{H}^{d_1})$ and $(\mathbb{R}^n, \mathcal{M}^{d_2}, \mathcal{H}^{d_2})$ are isomorphic.

Here $\mathcal{M}^d$ denotes the σ-algebra of measurable sets with respect to $\mathcal{H}^d$.
But no in *ZFC*.

Theorem (A. Máthé)

Let $0 < d_1 < d_2 < n$. Then the measure spaces $(\mathbb{R}^n, \mathcal{B}, \mathcal{H}^{d_1})$ and $(\mathbb{R}^n, \mathcal{B}, \mathcal{H}^{d_2})$ are not isomorphic.

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Measurable Sierpiński sets

Definition

- A set $S \subset \mathbb{R}^2$ is a Sierpiński set if all of its horizontal sections are countable and all of its vertical sections are co-countable.
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Theorem (M.E.)

Let $0 < d < 2$. Then there are no $\mathcal{H}^d$-measurable Sierpiński sets.

Theorem (Fremlin)

($\mathrm{add}(\mathcal{N}) = \mathfrak{c}$) There exists an $\mathcal{H}^1$-measurable Sierpiński set in the sense of measure.

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Question

Is it consistent that there exists a Sierpiński set in the sense of measure but no $\mathcal{H}^1$-measurable ones exist?

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Let $\mathcal{A}$ be a σ-algebra of subsets of a set X. A set $H \subset X$ small with respect to $\mathcal{A}$ if every subset of H belongs to $\mathcal{A}$. A set $A \in \mathcal{A}$ is a measurable hull of $H \subset X$ with respect to $\mathcal{A}$ if $H \subset A$ and for every $B \in \mathcal{A}$ such that $H \subset B \subset A$ the set $A \setminus B$ is small.

Remark

For example it is not hard to see that if $\mathcal{A}$ is the Borel, Lebesgue or Baire σ-algebra in $\mathbb{R}^n$, then the small sets are the countable, Lebesgue negligible and first category sets, respectively. One can also prove that with respect to the Lebesgue or Baire σ-algebra, every subset of $\mathbb{R}^n$ has a measurable hull, while in the case of the Borel sets this is not true. What makes these notions interesting is a theorem of Szpilrajn-Marczewski, asserting that if every subset of X has a measurable hull, then $\mathcal{A}$ is closed under the Souslin operation.

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Definition

A subset X of a Polish group G is called Haar null if there exists a Borel set $B \supset X$ and Borel probability measure μ on G such that $\mu(gBg') = 0$ for every $g, g' \in G$.

This definition is justified by the following theorem.

Theorem (Christensen)

A subset of a locally compact Polish group is Haar null in the above sense iff it is of Haar measure zero.

There has been quite some interest in this notion among set theorists lately. Problem FC on Fremlin's list basically asks: "But why do we need this Borel set B?" He actually proposed the real line as a possible example.

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Let $X \subset \mathbb{R}$, and let λ denote Lebesgue measure.

$$\lambda(X) = 0 \iff \exists \mu \text{ Borel probability measure s.t. } \mu(X + t) = 0 \ (\forall t \in \mathbb{R})?$$

Remark

Fremlin remarked that the answer is in the negative under *CH*.

Theorem (M.E.-Steprāns)

Let $K \subset \mathbb{R}$ be a compact set with $\dim_p K < 1/2$. Then there exists $X \subset \mathbb{R}$ with $\lambda(X) > 0$ such that $|K \cap (X + t)| \leq 1$ for every $t \in \mathbb{R}$.

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Question (Zapletal)

Are all forcing notions considered in the monograph "Forcing Idealized" homogeneous?

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If $\mathcal{I}$ is the σ-ideal of subsets of σ-finite $\mathcal{H}^{\frac{1}{2}}$-measure of the real line then $\mathbb{P}_{\mathcal{I}}$ is not homogeneous.

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Covering $\mathbb{R}$ with few translates of a compact nullset

Theorem (Gruenhage)

If C is the classical triadic Cantor set and $C + T = \mathbb{R}$ then $|T| = \mathfrak{c}$.

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Can we replace C by an arbitrary compact nullset?

This is of course true under CH, so the question asks if this holds in *ZFC*.
As there was no progress for a while, Mauldin asked the following.

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What if $\dim_H C < 1$?

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Let κ be a cardinal. Suppose that κ many translates of a suitable compact nullset cover $\mathbb{R}^2$. Is this then true in $\mathbb{R}$?

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Is there an ordering of the plane such that every initial segment is $\mathcal{H}^1$-null?

They noted that under CH the answer is affirmative.

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It is consistent that there is no such ordering.

The proof is a forcing construction showing that $\mathrm{cov}(\mathcal{N}^1) = \omega_2 \wedge \mathrm{non}(\mathcal{N}^1) = \omega_1$ is consistent and implies that there is no such ordering.

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Is there an ordering of the plane such that every initial segment is $\mathcal{H}^1$-null?

They noted that under CH the answer is affirmative.

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