The basis problem (for compacta satisfying high separation axioms)

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Outline

Basis problems in combinatorial set theory

An elegant and deep program set forth by Stevo Todorcevic.

-S. Todorcevic, *Basis problems in combinatorial set theory*, Proceedings of the International Congress of Mathematicians, Vol. II (Berlin, 1998), Doc. Math. 1998, Extra Vol. II, 43–52.

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Starting out with a class of structures $\mathcal{S}$,

1. Recognize the critical members of $\mathcal{S}$.
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The 3-element basis conjecture

For $\mathcal{S}$ a class of topological spaces, and "complete" interpreted in terms of topological embedding, a basis for $\mathcal{S}$ is then a subclass $\mathcal{S}_0$ such that each member of $\mathcal{S}$ contains a homeomorphic copy of a member of $\mathcal{S}_0$.

Denote by

- $D(\omega_1)$, the discrete space on ω_1,
- B, an uncountable subspace of the unit interval, and
- $B \times \{0\}$, B considered as a subspace of the split interval.

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Subspaces of monotonically normal compacta

- The main tool here is Nikiel's conjecture, proved by M.E. Rudin: a Hausdorff space is an image of a compact linearly ordered space iff it is monotonically normal.
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A space is perfectly normal if it is normal and every closed subset is a G_δ-set. Recall that in the setting of compact spaces, HL and perfect normality coincide.

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Definition

An *embedded Aronszajn compactum* is a closed subspace $X \subseteq [0,1]^{\omega_1}$ with $w(X) = \aleph_1$ and $\chi(X) = \aleph_0$ such that for some club $C \subseteq \omega_1$: for each $\alpha \in C, \mathcal{L}_\alpha := \{x \in X_\alpha : |(\sigma_\alpha^{\omega_1})^{-1}\{x\}| > 1\}$ is countable.

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For each such X, define $T = T(X) := \bigcup\{L_\alpha : \alpha \in C\}$, and let $\triangleleft$ denote the following order: if $\alpha, \beta \in C, \alpha < \beta, x \in \mathcal{L}_\alpha$ and $y \in \mathcal{L}_\beta$ then $x \triangleleft y$ iff $x = \pi_\alpha^\beta(y)$. $(T, \triangleleft)$ is then an $\aleph_1$-tree.

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Let X be an HL compact space. Then either X has a quotient which is an Aronszajn compactum, or X is a premetric compactum of degree at most 2.

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Some theorems which shed light on the main problem at hand.

Avraham, Shelah [PFA]

Every two Aronszajn trees are club-isomorphic.