Rapid ultrafilters and summable ideals

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Rapid ultrafilters

Definition.

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Theorem (Booth?).

(CH) Rapid ultrafilters exist.

Theorem (Miller).

In Laver's model there are no rapid ultrafilters.

Summable ideals

Definition.

Given a function $g : \mathbb{N} \to [0, \infty)$ such that $\sum_{n\in\mathbb{N}} g(n) = +\infty$ then the family

$$\mathcal{I}_g = \{A \subseteq \mathbb{N} : \sum_{a\in A} g(a) < +\infty\}$$

is a proper ideal which we call summable ideal determined by function g.

A summable ideal is tall (dense) if and only if $\lim_{n\to\infty} g(n) = 0$.

Characterization of rapid ultrafilters

Theorem (Vojtáš).

The following are equivalent for an ultrafilter $\mathcal{U} \in \omega^*$:

- $\mathcal{U}$ is rapid
- $\mathcal{U} \cap \mathcal{I}_g \neq \emptyset$ for every tall summable ideal $\mathcal{I}_g$

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One can add two more equivalent conditions:

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- $(\forall f : \omega \to \mathbb{N}$ finite-to-one$)$ $(\exists U \in \mathcal{U})$ such that $f[U] \in \mathcal{I}_g$ for every tall summable ideal $\mathcal{I}_g$
(= $\mathcal{U}$ is a weak $\mathcal{I}_g$-ultrafilter for every $\mathcal{I}_g$)

$\mathcal{I}_g$-ultrafilters

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An ultrafilter $\mathcal{U} \in \omega^*$ is called an $\mathcal{I}_g$-ultrafilter if for every $f : \omega \to \mathbb{N}$ there exists $U \in \mathcal{U}$ such that $f[U] \in \mathcal{I}_g$.

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Theorem 1.

($\mathrm{MA}_{\mathrm{ctble}}$) There exists $\mathcal{U} \in \omega^*$ such that $\mathcal{U}$ is an $\mathcal{I}_g$-ultrafilter for every tall summable ideal $\mathcal{I}_g$.

Rapid ultrafilters vs. $\mathcal{I}_g$-ultrafilters

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Theorem 2.

($\mathrm{MA}_{\mathrm{ctble}}$) There is a rapid ultrafilter which is not an $\mathcal{I}_g$-ultrafilter for any summable ideal $\mathcal{I}_g$.

$\mathcal{I}_g$-ultrafilters vs. rapid ultrafilters

If $\mathcal{U} \in \omega^*$ is an $\mathcal{I}_g$-ultrafilter for every tall summable ideal $\mathcal{I}_g$ then $\mathcal{U}$ is a rapid ultrafilter.

$\mathcal{I}_g$-ultrafilters vs. rapid ultrafilters

If $\mathcal{U} \in \omega^*$ is an $\mathcal{I}_g$-ultrafilter for every tall summable ideal $\mathcal{I}_g$ then $\mathcal{U}$ is a rapid ultrafilter.

Theorem 3.

($\mathrm{MA}_{\mathrm{ctble}}$) There is an $\mathcal{I}_{\frac{1}{n}}$-ultrafilter which is not a rapid ultrafilter.

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($\mathrm{MA}_{\mathrm{ctble}}$) There is an $\mathcal{I}_{\frac{1}{n}}$-ultrafilter which is not a rapid ultrafilter.

Theorem 4.

($\mathrm{MA}_{\sigma-\mathrm{centered}}$) For every tall summable ideal $\mathcal{I}_g$ there is an $\mathcal{I}_g$-ultrafilter which is not rapid.

Possible extension and its limits

Is it possible that an ultrafilter is an $\mathcal{I}_g$-ultrafilter for “many" tall summable ideals simultaneously and still not a rapid ultrafilter?

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Proposition 5.

There is a family $\mathcal{D}$ of tall summable ideals such that $|\mathcal{D}| = \mathfrak{d}$ and an ultrafilter $\mathcal{U} \in \omega^*$ is rapid if and only if it has a nonempty intersection with every tall summable ideal in $\mathcal{D}$.

Possible extension and its limits

Proposition 6.

If $\mathcal{D}$ is a family of tall summable ideals and $|\mathcal{D}| < \mathfrak{b}$ then there exists a tall summable ideal $\mathcal{I}_g$ such that $\mathcal{I}_g \subseteq \mathcal{I}_h$ for every $\mathcal{I}_h \in \mathcal{D}$.

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Proposition 6.

If $\mathcal{D}$ is a family of tall summable ideals and $|\mathcal{D}| < \mathfrak{b}$ then there exists a tall summable ideal $\mathcal{I}_g$ such that $\mathcal{I}_g \subseteq \mathcal{I}_h$ for every $\mathcal{I}_h \in \mathcal{D}$.

Corollary 7.

$(MA_{\sigma-\text{centered}})$ If $\mathcal{D}$ is a family of tall summable ideals and $|\mathcal{D}| < \mathfrak{d}$ then there exists an ultrafilter $\mathcal{U} \in \omega^*$ such that $\mathcal{U}$ is an $\mathcal{I}$-ultrafilter for every $\mathcal{I} \in \mathcal{D}$, but $\mathcal{U}$ is not a rapid ultrafilter.

$\mathcal{I}_g$-ultrafilters need not be rapid

Proof of Theorem 4.

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Proposition 4a.

For every tall summable ideal $\mathcal{I}_g$ there is a tall summable ideal $\mathcal{I}_h$ such that $\mathcal{I}_g \not\leq_K \mathcal{I}_h$.

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Proposition 4a.

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Theorem 4b.

($MA_{\sigma-\text{centered}}$) For arbitrary tall summable ideals $\mathcal{I}_g$ and $\mathcal{I}_h$ such that $\mathcal{I}_g \not\leq_K \mathcal{I}_h$ there is an $\mathcal{I}_g$-ultrafilter $\mathcal{U}$ with $\mathcal{U} \cap \mathcal{I}_h = \emptyset$.

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Theorem 4b.

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Theorem 4b.

($\text{MA}_{\sigma-\text{centered}}$) For arbitrary tall summable ideals $\mathcal{I}_g$ and $\mathcal{I}_h$ such that $\mathcal{I}_g \not\leq_K \mathcal{I}_h$ there is an $\mathcal{I}_g$-ultrafilter $\mathcal{U}$ with $\mathcal{U} \cap \mathcal{I}_h = \emptyset$.

1. Enumerate all functions in ${}^\omega\omega$ as $\{f_\alpha : \alpha < \mathfrak{c}\}$.

2. For $\alpha < \mathfrak{c}$ construct filter bases $\mathcal{F}_\alpha$ such that for every $\alpha < \mathfrak{c}$:
   - (i) $\mathcal{F}_0$ is the Fréchet filter
   - (ii) $\mathcal{F}_\alpha \supseteq \mathcal{F}_\beta$ whenever $\alpha \geq \beta$
   - (iii) $\mathcal{F}_\gamma = \bigcup_{\alpha<\gamma} \mathcal{F}_\alpha$, for γ limit
   - (iv) $(\forall\alpha)\ |\mathcal{F}_\alpha| \leq |\alpha + 1| \cdot \omega$
   - (v) $(\forall\alpha)\ \mathcal{F}_\alpha \cap \mathcal{I}_h = \emptyset$
   - (vi) $(\forall\alpha)\ (\exists F \in \mathcal{F}_{\alpha+1})\ f_\alpha[F] \in \mathcal{I}_g$

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Proof of Theorem 4. — more details

Lemma 4c.

($MA_{\sigma-\text{centered}}$) Assume $\mathcal{I}_g$ and $\mathcal{I}_h$ are two tall summable ideals such that $\mathcal{I}_g \not\leq_K \mathcal{I}_h$. Assume $\mathcal{F}$ is a filter base with $|\mathcal{F}| < \mathfrak{c}$ such that $\mathcal{F} \cap \mathcal{I}_h = \emptyset$ and a function $f \in \omega^\omega$ is given. Then there exists $G \subseteq \omega$ such that $f[G] \in \mathcal{I}_g$ and $G \cap F \notin \mathcal{I}_h$ for every $F \in \mathcal{F}$.

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Lemma 4c.

($MA_{\sigma-\text{centered}}$) Assume $\mathcal{I}_g$ and $\mathcal{I}_h$ are two tall summable ideals such that $\mathcal{I}_g \not\leq_K \mathcal{I}_h$. Assume $\mathcal{F}$ is a filter base with $|\mathcal{F}| < \mathfrak{c}$ such that $\mathcal{F} \cap \mathcal{I}_h = \emptyset$ and a function $f \in \omega^\omega$ is given. Then there exists $G \subseteq \omega$ such that $f[G] \in \mathcal{I}_g$ and $G \cap F \notin \mathcal{I}_h$ for every $F \in \mathcal{F}$.

Lemma 4d.

($MA_{\sigma-\text{centered}}$) Assume $\mathcal{I}_h$ is a tall summable ideal and $\mathcal{F}$ is a filter base with $|\mathcal{F}| < \mathfrak{c}$ such that $\mathcal{F} \cap \mathcal{I}_h = \emptyset$. Then there exists a set $H \subseteq \omega$ such that $H \notin \mathcal{I}_h$ and $H \setminus F$ is finite for every $F \in \mathcal{F}$.

$\mathcal{I}_g$-ultrafilters need not be rapid

Proof of Theorem 4. — more details

Lemma 4e.

Assume $f \in \omega^\omega$, $\mathcal{I}_g$ and $\mathcal{I}_h$ are tall summable ideals with $\mathcal{I}_g \not\leq_K \mathcal{I}_h$. If H is an infinite subset of ω such that $H \notin \mathcal{I}_h$ and $f[H] \notin \mathcal{I}_g$ then there exists $A \subseteq f[H]$ such that $A \in \mathcal{I}_g$ and $f^{-1}[A] \cap H \notin \mathcal{I}_h$.

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Lemma 4f.

Assume $\mathcal{I}_g$ is a tall summable ideal determined by a decreasing function g, A is a subset of ω and $B \subseteq A$. Then

1. $A \in \mathcal{I}_g$ if and only if $A + 1 \in \mathcal{I}_g$
2. $A \in \mathcal{I}_g$ if and only if $B + 1 \cup (A \setminus B) \in \mathcal{I}_g$

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