Definable graphs of low complexity

Stefan Geschke

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Definable graphs

Definition

A *graph* is a set V of vertices together with a set $E \subseteq [V]^2$ of edges.

If $G = (V, E)$ is a graph whose set V of vertices carries a topology, then G is open, closed, Borel, analytic, ... if the *edge-relation* $\{(x, y) \in V^2 : \{x, y\} \in E\}$ of G has the respective property as a subset of $V^2 \setminus \{(v, v) : v \in V\}$.

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Cardinal invariants

Definition

Let $G = (X, E)$ be a graph. Then $A \subseteq X$ is a *G-clique* (a *clique* in G) if $[A]^2 \subseteq E$.

$A \subseteq X$ is *G-independent* (an *independent* set in G) if $[A]^2 \cap E = \emptyset$. (Independent sets are sometimes called discrete.)

$A \subseteq X$ is *G-homogeneous* (a *homogeneous* set in G) if A is either independent or a clique in G.

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Definition

The *clique-number* of a graph G is the supremum of the sizes of all G-cliques.

Clique-numbers are degenerate for graphs of low complexity:

Theorem (Kubiś)

A G_δ-graph with an uncountable clique has a perfect clique.

This is sharp: there is an F_σ-graph on 2^ω with a clique of size $\aleph_1$ but no perfect clique. The graph is a variant of the symmetrization of Turing reducibility (Folklore, Kubiś-Shelah, Mátrai).

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Definition

The *chromatic number* of a graph G is the least size of a family of G-independent (G-discrete) sets that covers all the vertices of G.

The chromatic number of open graphs is degenerate in the following sense: An open graph is either countably chromatic or has a perfect clique and hence chromatic number $2^{\aleph_0}$ (provable instance of Todorcevic's OCA).

This dichotomy fails for closed graphs.

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This dichotomy fails for closed graphs.

Definition

The *cochromatic number* of a graph $G = (V, E)$ is the least cardinality of a family of homogeneous sets that covers V.

Theorem

a) There is a clopen graph $G_{\min}$ on 2^ω such that a clopen graph G on a Polish space has an uncountable cochromatic number iff $G_{\min}$ embeds into G (GKKS).

b) There is a clopen graph $G_{\max}$ on 2^ω whose cochromatic number is maximal among all cochromatic numbers of clopen graphs on Polish spaces (GGK).

c) It is consistent that the cochromatic number of $G_{\max}$ is $\aleph_1 < 2^{\aleph_0}$ (GKKS).

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Definition

For any graph G let $\mathrm{Age}(G)$ denote the class of finite graphs that embed into G.

Theorem

Let G be a clopen graph on a Polish space. If $\mathrm{Age}(G)$ is generated by a finite set of finite graphs by taking isomorphic copies, induced subgraphs, and substitution, then the cochromatic number of G is countable or equal to the cochromatic number of G_{min}.

Example

$\mathrm{Age}(G_{\mathrm{min}})$ is generated by two graphs with two vertices:
The edge and the non-edge.

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Clopen graphs and continuous colorings

We first observe that a clopen graph G on a Hausdorff space X is the same as a coloring $c : [X]^2 \to 2$ that is continuous wrt the natural topology on $[X]^2$, by identifying the set of edges of G with its characteristic function.

Recall that every Polish space is the 1-1 continuous image of a closed subset of ω^ω.

Hence every clopen graph on a Polish space can be pulled back to a combinatorially isomorphic clopen graph on a closed subset of ω^ω.

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Definition

Let X be a closed subset of ω^ω. For distinct $x, y \in X$ let $\Delta(x, y)$ be the least $m \in \omega$ with $x(m) \neq y(m)$.

A continuous coloring $c : [X]^2 \to 2$ is *uniformly continuous* there is a function $f : \omega \to \omega$ such that for all $m \in \omega$ and distinct $x, y \in X$ with $\Delta(x, y) = m$, then $c(x, y)$ only depends on $x \restriction f(m)$ and $y \restriction f(m)$.

The continuous coloring c is of *depth k* if for all $x, y \in X$ with $x \neq y$, $c(x, y)$ only depends on $x \restriction (\Delta(x, y) + k)$ and $y \restriction (\Delta(x, y) + k)$.

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Theorem (GGK)

There is a universal continuous coloring $c : [\omega^\omega]^2 \to 2$ *of depth* 1.

Lemma (GGK)

Let $X \subseteq \omega^\omega$ *be closed and* $c : [X]^2 \to 2$ *uniformly continuous. Then* c *is topologically isomorphic to a continuous coloring* $d : [Y]^2 \to 2$ *of depth* 2 *on a closed subset of* ω^ω.

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There is a universal continuous coloring $c : [\omega^\omega]^2 \to 2$ *of depth* 2. *This coloring is also universal for uniformly continuous colorings on* ω^ω.

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Saturation

Definition

A *type* over a set A is a function $f : A \to 2$.

If $c : [X]^2 \to 2$ is a coloring and f is a type over a set $A \subseteq X$, then $x \in X \setminus A$ *realizes* f if for all $a \in A$, $c(a, x) = f(a)$.

For a cardinal κ, the coloring c is *κ-saturated* if all types over subsets A of X of size $< \kappa$ are realized.

Lemma

Let $X \subseteq \omega^\omega$ be closed and let $c : [X]^2 \to 2$ be of depth 2. Let $A \subseteq X$ be such that all types over 3-element subsets of A are realized in X. Let $y, z \in A$ be distinct and $n = \Delta(y, z)$.

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Corollary

If $X \subseteq \omega^\omega$ is closed and $c : [X]^2 \to 2$ is of depth 2, then for every uncountable set $A \subseteq X$ there is a type over a 3-element subset of A that is not realized in X.

Example

There is an $\aleph_0$-saturated clopen graph on ω^ω.

Corollary

No uniformly continuous coloring is universal for all continuous colorings on ω^ω.

Theorem

No $\aleph_1$-saturated graph embeds into a clopen graph on any Polish space.

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Example

a) There is an F_σ-graph on 2^ω that is $\aleph_1$-saturated and has no perfect cliques. It has perfect independent sets and the chromatic number is $\aleph_1$. (Saturation pointed out by Conley, chromatic number by Mátrai)

b) X be either the Cantor space 2^ω or the Baire space ω^ω. Let $\alpha \in \omega_1 \setminus \{0\}$ and $n \in \omega \setminus \{0\}$. Let Γ be one of the following classes of subsets of $X^2 \setminus \{(x,x) : x \in X\}$: $\mathbf{\Sigma}^0_\alpha$, $\mathbf{\Pi}^0_\alpha$, $\mathbf{\Sigma}^1_n$, and $\mathbf{\Pi}^1_n$. Then there is a graph G on X in the class Γ such that every graph on X in the class Γ embeds into G by a topological embedding. (Pointed out by B. Miller)

Example

a) There is an F_σ-graph on 2^ω that is $\aleph_1$-saturated and has no perfect cliques. It has perfect independent sets and the chromatic number is $\aleph_1$. (Saturation pointed out by Conley, chromatic number by Mátrai)

b) X be either the Cantor space 2^ω or the Baire space ω^ω. Let $\alpha \in \omega_1 \setminus \{0\}$ and $n \in \omega \setminus \{0\}$. Let Γ be one of the following classes of subsets of $X^2 \setminus \{(x,x) : x \in X\}$: $\mathbf{\Sigma}^0_\alpha$, $\mathbf{\Pi}^0_\alpha$, $\mathbf{\Sigma}^1_n$, and $\mathbf{\Pi}^1_n$. Then there is a graph G on X in the class Γ such that every graph on X in the class Γ embeds into G by a topological embedding. (Pointed out by B. Miller)

The compact case

Lemma

Let X be a compact metric space and let $c : [X]^2 \to 2$ be continuous. Let $\mathrm{Comp}(X)$ denote the space of connected components of X with the quotient topology.

Then $\mathrm{Comp}(X)$ is compact, zero-dimensional, and metric. In particular, $\mathrm{Comp}(X)$ embeds into 2^ω.

If $c : [X]^2 \to 2$ is continuous, then c is constant on each connected component of X and induces a continuous coloring on $\mathrm{Comp}(X)$.

This lemma shows that we should study continuous colorings on 2^ω or on compact subspaces of ω^ω to obtain information about the class of all clopen graphs on compact metric spaces.

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Definition

Let $(P, \leq_P)$ and $(Q, \leq_Q)$ be directed sets. A map $\varphi : P \to Q$ is *Tukey* if for all $q \in Q$ there is $p \in P$ such that for all $x \in P$, $\varphi(x) \leq_Q q$ implies $x \leq_P p$. In other words, a map is *Tukey* if preimages of bounded sets are bounded.

If there is a Tukey map from P to Q we say that P is *Tukey-reducible* to Q. If P is Tukey reducible to Q and Q is Tukey reducible to P, then P and Q are *Tukey-equivalent*.

Definition

Let $\mathcal{C}$ be the set of clopen graphs on 2^ω, let $\leq_t$ denote topological embeddability, and $\leq_c$ denote combinatorial embeddability between graphs in $\mathcal{C}$.

Our goal is to prove the Tukey eqivalence of the directed sets $(\omega^\omega, \leq^*)$, $(\mathcal{C}, \leq_t)$, and $(\mathcal{C}, \leq_c)$.

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Lemma

Let X be a compact subset of ω^ω and let $c : [X]^2 \to 2$ be continuous.
Then c is uniformly continuous.

Corollary

Every continuous coloring on a compact, zero-dimensional, metric space embeds into the universal coloring of depth 2 on ω^ω.

This suggests a way of assigning to each clopen graph G on 2^ω a function $f : \omega \to \omega$:
We assume that G is an induced subgraph of the graph corresponding to the universal coloring of depth 2 on ω^ω. Now let $f : \omega \to \omega$ be a function is that coordinate wise an upper bound of all vertices of G.

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Lemma

To every function $f : \omega \to \omega$ we can assign a clopen graph G_f on 2^ω such that whenever G_f embeds combinatorially into a graph of depth 2 on a closed subset X of ω^ω, then for all but finitely many $n \in \omega$, $|X \restriction n| > f(n)$.

Theorem

The directed sets $(\omega^\omega, \leq^*)$, $(\mathcal{C}, \leq_t)$, and $(\mathcal{C}, \leq_c)$ are Tukey equivalent.
In particular

$$\mathfrak{d}(\mathcal{C}, \leq_t) = \mathfrak{d}(\mathcal{C}, \leq_c) = \mathfrak{d}$$

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We can extend some of the previously used methods to larger cardinals:

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If G is a clopen graph on a compact space and A is an infinite set of vertices of G, then there is a type over a 3-element subset of A that is not realized in G.
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Thank you!