Covering properties of ideals

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joint work with Marek Balcerzak and Barnabás Farkas

Theorem (Elekes)

Let $(A_n)_{n\in\omega}$ be a sequence of measurable sets that covers μ-almost every $x \in [0,1]$ infinitely many times. Then there exists a set $M \subseteq \omega$ of asymptotic density zero such that $(A_n)_{n\in M}$ also covers μ-almost every $x \in [0,1]$ infinitely mamy times.

Corollary

The density zero ideal is random-indestructible.

$\mathcal{J}$ *is* $\mathbb{P}$*-indestructible* if $\Vdash_{\mathbb{P}} \exists\, A \in \mathcal{J}\ |\dot{X} \cap A| = \aleph_0$ for each $\mathbb{P}$-name $\dot{X}$ for an infinite subset of ω, i.e. in $V^{\mathbb{P}}$ the ideal generated by $\mathcal{J}$ is tall.

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Definition

A sequence $(A_n)_{n\in\omega}$ is an *I-a.e. infinite-fold cover of X* if

$$\big\{x \in X : \{n \in \omega : x \in A_n\} \text{ is finite}\big\} \in I \ , \quad \text{i.e.} \quad \limsup_{n\in\omega} A_n \in I^*.$$

Let $\mathcal{A}$ be a σ-algebra of subsets of X and an ideal $\mathcal{J}$ on ω. The pair $(\mathcal{A}, I)$ has the *$\mathcal{J}$-covering property* if for every I-a.e. infinite-fold cover $(A_n)_{n\in\omega}$ of X by sets from $\mathcal{A}$, there is a set $S \in \mathcal{J}$ such that $(A_n)_{n\in S}$ is also an I-a.e. infinite-fold cover of X.

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$I_1 \subseteq I_2 \Rightarrow \mathrm{CP}(I_1) \subseteq \mathrm{CP}(I_2)$

Elekes' theorem says that $\mathcal{Z} \in \mathrm{CP}(\mathcal{N})$.

If $(\mathcal{A}, I)$ has the $\mathcal{J}$-covering property, then $\mathrm{non}^*(\mathcal{J}) > \omega$.

$\mathrm{non}^*(\mathcal{J}) = \min\left\{|\mathcal{H}| : \mathcal{H} \subseteq [\omega]^\omega \text{ and } \nexists\, A \in \mathcal{J}\ \forall\, H \in \mathcal{H}\ |A \cap H| = \omega\right\}$.

If $\mathcal{J}_0 \leq_{\mathrm{KB}} \mathcal{J}_1$ and $(\mathcal{A}, I)$ has the $\mathcal{J}_0$-covering property, then $(\mathcal{A}, I)$ has the $\mathcal{J}_1$-covering property as well.
$\mathcal{J}_0 \leq_{\mathrm{KB}} \mathcal{J}_1$ iff there is a finite-to-one function $f : \omega \to \omega$ such that $f^{-1}[A] \in \mathcal{J}_1$ for each $A \in \mathcal{J}_0$.

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$\mathcal{N}$ does not have the $\mathcal{I}_{1/n}$-covering property.

there is no ideal $\mathcal{J}$ on ω such that I has the $\mathcal{J}$-covering property if $I = [\omega^\omega]^{\leq\omega}$, NWD (the ideal of nowhere dense subsets of ω^ω), or $\mathcal{K}_\sigma$.

Fubini product of ideals $I \subseteq \mathcal{P}(X)$ and $K \subseteq \mathcal{P}(Y)$:

$$I \otimes K = \left\{A \subseteq X \times Y : \{x \in X : (A)_x \in K\} \in I^*\right\}.$$

Theorem

$\mathrm{CP}_{P\text{-}ideals}(\mathcal{N} \otimes \mathcal{M}) = \mathrm{CP}_{P\text{-}ideals}(\mathcal{N})$.

Question

Is the analogous result true for $\mathcal{M} \otimes \mathcal{N}$?

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$\mathbb{P}_I = \mathrm{Borel}(X) \setminus I$ ordered by inclusion where X is a Polish space and I is a σ-ideal on X (with Borel base).

Theorem

Let I be a σ-ideal on a Polish space X, and assume that $\mathbb{P}_I$ is proper. If I has the $\mathcal{J}$-covering property, then $\mathcal{J}$ is $\mathbb{P}_I$-indestructible.

Example

$\mathcal{ED}$ and $\mathrm{Fin} \otimes \mathrm{Fin}$ are Cohen-indestructible but $\mathcal{M}$ does not have the $\mathcal{ED}$- or $\mathrm{Fin} \otimes \mathrm{Fin}$-covering properties.

$$\mathcal{ED} = \left\{ A \subseteq \omega \times \omega : \limsup_{n \to \infty} |(A)_n| < \infty \right\}$$

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Theorem

Assume $\mathcal{J}$ is a Borel ideal. Then $\mathcal{M}$ has the $\mathcal{J}$-covering property iff $\mathcal{J}$ is not a weak Q-ideal (i.e. $\mathcal{ED}_{\mathrm{fin}} \leq_{\mathrm{KB}} \mathcal{J}$). In other words,

$$\mathrm{CP}_{\mathrm{Borel}}(\mathcal{M}) = \{\mathcal{J} : \mathcal{J} \text{ is a Borel non weak Q-ideal}\}.$$

$\mathcal{J}$ on ω is called *weak Q-ideal* if for each partition $(P_n)_{n\in\omega}$ of ω into finite sets, there is an $X \in \mathcal{J}^+$ such that $|X \cap P_n| \leq 1$ for each n.

$\mathcal{ED}_{\mathrm{fin}} = \mathcal{ED} \restriction \Delta$ where $\Delta = \{(n,m) \in \omega \times \omega : m \leq n\}$.

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$$\mathcal{Z}_u = \left\{ A \subseteq \omega : \lim_{n\to\infty} \frac{S_n(A)}{n} = 0 \right\}.$$

$$S_n(A) = \max\left\{ |A \cap [k, k+n)| : k \in \omega \right\},$$

Theorem

$\mathcal{Z} \not\leq_{\mathrm{KB}} \mathcal{Z}_u$ and $\mathcal{N}$ has the $\mathcal{Z}_u$-covering property.($\mathcal{Z}$ is not KB-minimal in $\mathrm{CP}_{\mathrm{Borel}}(\mathcal{N})$)

Question

Does there exist a KB-smallest (or at least KB-minimal) element is $\mathrm{CP}_{\mathrm{Borel}}(\mathcal{N})$?

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$\mathcal{Z} \not\leq_{\mathrm{KB}} \mathcal{Z}_u$ and $\mathcal{N}$ has the $\mathcal{Z}_u$-covering property.($\mathcal{Z}$ is not KB-minimal in $\mathrm{CP}_{\mathrm{Borel}}(\mathcal{N})$)

Question

Does there exist a KB-smallest (or at least KB-minimal) element is $\mathrm{CP}_{\mathrm{Borel}}(\mathcal{N})$?

Theorem

Assume $\mathfrak{t} = \mathfrak{c}$ and $|\mathcal{A}| \leq \mathfrak{c}$. Then there is no KB-smallest element in $\mathrm{CP}(\mathcal{A}, I)$.

Theorem

After adding ω_1 Cohen-reals, there is an ideal $\mathcal{J}$ such that $\mathcal{ED}_{\mathrm{fin}} \not\leq_{\mathrm{KB}} \mathcal{J}$ (in particular, $\mathcal{Z}_u \not\leq_{\mathrm{KB}} \mathcal{J}$) but $\mathcal{N}$ and $\mathcal{M}$ have the $\mathcal{J}$-covering property.

Question

Is it provable in **ZFC** that there are no KB-smallest elements of $\mathrm{CP}(\mathcal{N})$ and $\mathrm{CP}(\mathcal{M})$? Or at least, is it provable that $\mathcal{Z}_u$ and $\mathcal{ED}_{\mathrm{fin}}$ are not the KB-smallest members of these families?

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Theorem

Assume that I is a translation invariant ccc σ-ideal on $\mathbb{R}$ fulfilling the condition

$$\mathbb{Q} + A \in I^* \text{ for each } A \in \mathrm{Borel}(\mathbb{R}) \setminus I.$$

Fix a P-ideal $\mathcal{J}$ on ω. If I does not have the $\mathcal{J}$-covering property, then there exists an infinite-fold Borel cover $(A'_n)_{n\in\omega}$ of $\mathbb{R}$ with $\limsup_{n\in S} A'_n \in I$ for all $S \in \mathcal{J}$.