Are Eberlein-Grothendieck scattered spaces σ-discrete?

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Definitions

Unless otherwise stated, every topological space in this presentation is assumed to be Tychonoff. The set of real numbers with the natural topology is denoted by $\mathbb{R}$. For a space X the family of all open subsets of X is denoted by $\tau(X)$ and the family of all compact subspaces of X is denoted by $\mathcal{K}(X)$. A space X is scattered if for every non void $Y \subset X$ there is $y \in Y$ such that $\{y\} \in \tau(Y)$. For a space X denote by $X^{(0)}$ the set of the isolated points of X. If $X^{(\alpha)}$ is defined for any $\alpha < \gamma$, let $X^{(\gamma)}$ be the set of isolated points of $X \setminus \bigcup\limits_{\alpha<\gamma} X^{(\alpha)}$. The set $X^{(\gamma)}$ is called the γ-th scattering level of the space X.

Definitions

It is clear that a scattered space is the union of its scattering levels. The height of a scattered space is the first ordinal κ for which $X^{(\kappa)} = \varnothing$. A transfinite sequence $\{x_\alpha : \alpha < \lambda\}$ of elements of a space X is right-separated if for every $\mu < \lambda$ there is $U \in \tau(X)$ such that $U \cap \{x_\alpha : \alpha < \lambda\} = \{x_\alpha : \alpha < \mu\}$. A space X is scattered if and only if it can be written as a right separated sequence $X = \{x_\alpha : \alpha < \lambda\}$.

Definitions

The space of all continuous functions from a space X into a space Y, endowed with the topology inherited from the product space Y^X, is denoted by $C_p(X, Y)$. On the other hand, $C_u(X)$ is the space of all continuous real-valued functions on a space X, with the topology of uniform convergence.

Definitions

Arhangel'skii defined Eberlein-Grothendieck spaces as those homeomorphic to a subspace of $C_p(K)$ for some compact space K. Notice that if X is a subset of a Banach space E with the weak topology then X embeds in $C_p(\mathbf{B}X^*)$ hence X is Eberlein-Grothendieck.

Answer to Benyamini, Rudin, Simon & Wage

Theorem (Alster)

For any compact space K TFAE:

- *K is σ-discrete and Corson;*
- *K is scattered and Corson;*
- *K is strong Eberlein.*

What if X is not compact? (Arhangel'skii?)

Particular case (Haydon)

Assuming K is compact, is $C_p(K, \{0, 1\})$ σ-discrete whenever it is σ-scattered?

Problem 1

Are Eberlein-Grothendieck scattered spaces σ-discrete?

Relevance

The property SLD (JNR)

Given a space X, a metric ρ on X and $\varepsilon > 0$, a family $\mathcal{A}$ of subsets of X is ε-small if $diam_\rho(A) < \varepsilon$ for every $A \in \mathcal{A}$. A topological space (X, τ) has the property SLD with respect to a metric ρ on X if for every $\varepsilon > 0$ there is a countable cover $\{X_n : n \in \omega\}$ of X such that for each $n \in \omega$ the space X_n admits a τ-open cover which is ε-small.

σ-fagmentable spaces

A space (X, τ) is σ-fragmented by a metric ρ on X if for every $\varepsilon > 0$ there is a countable cover $\{X_n : n \in \omega\}$ of X such that for each $n \in \omega$ and every $Y \subset X_n$ there exists a nonempty $U \in \tau(Y)$ with $diam_\rho(U) < \varepsilon$.

Relevance

Kadets-Klee renorming

If a space has SLD with respect to some metric, then it is σ-fragmented as well, but the following is an open question: Are the properties of σ-fragmentability and SLD equivalent when X is a Banach space endowed with its weak topology and with its norm metric, or when X is of the form $C_p(K)$ endowed with the uniform metric?

Discrete version

If a space with the discrete metric is SLD then it is σ-discrete and if it is σ-fragmented then it is σ-scattered. Actually the following is still open: If $C_p(K, \{0, 1\})$ is scattered (respectively σ-discrete), does it imply that $C_p(K)$ is σ-fragmentable (respectively SLD)?

The case when $w(K) = \omega_1$ and $X \subset C_p(K)$

Theorem

If X is an Eberlein-Grothendieck locally compact scattered space of height lower than $\omega_1 \cdot \omega_1$, then X is σ-discrete.

Theorem

If X is an Eberlein-Grothendieck locally countable scattered space of cardinality ω_1, then X is σ-discrete.

Corollary

If $X = \{x_\alpha : \alpha < \omega_1\} \subset C_p(K)$ is a right-separated ω_1-sequence, then X is σ-discrete.

Furthermore

Corollary

Suppose $w(K) = \omega_1$, if $X \subset C_p(K)$ is locally countable separable and scattered then X is countable.

Corollary

If K is a scattered compact space of weight ω_1 then $C_p(K)$ is SLD if and only if for every $\varepsilon > 0$ there exists a family $\{X_n : n \in \omega\}$ that covers $C_p(K, \{0, 1\})$ and for each $n \in \omega$ the set X_n has an ε-open partitioning of length ω_1.

Proven by means of topological games

Theorem

Every Lindelöf Čech-complete scattered space is σ-compact.

Corollary

Every Eberlein-Grothendieck Lindelöf Čech-complete scattered space is σ-discrete.

A bad space

We applied the hereditary metalindelöfness of certain scattered spaces to prove they are σ-discrete. It is not very clear that this property implies σ-discreteness of scattered Tychonoff spaces. This is not true for general spaces as we can deduce from the following example.

Example

There exists a scattered space of class T_1 which is hereditarily meta-Lindelöf but is not σ-discrete.

Open problems, the Lindelöf case

- Is every Eberlein-Grothendieck Lindelöf scattered space σ-discrete?
- Suppose that L is a Lindelöf scattered space and K is a compact subspace of $C_p(L)$. Is $C_p(K)$ hereditarily weakly θ refinable?

The Čech complete case

We showed that Problem 1 has a positive partial answer for the case of Lindelöf Čech-complete scattered spaces. What if we remove the hypothesis that the space is Lindelöf?

- Is every Eberlein-Grothendieck Čech-complete scattered space σ-discrete?

Meta-Lindelöf spaces

- Is every Eberlein-Grothendieck hereditarily meta-Lindelöf scattered space σ-discrete?
- Let X be an Eberlein-Grothendieck hereditarily meta-Lindelöf scattered space of height and cardinality equal to ω_1. For every point $x \in X$ there is an open set U_x that isolates x in its scattering level. There is a point countable open refinement $\mathcal{V}$ of the cover $\{U_x : x \in X\}$. Let V_x be the intersection of U_x and the union of all the elements of $\mathcal{V}$ that contain x. Define the partially ordered set $\mathbb{P} = \{p \subset X : p \text{ is finite and } V_x \cap p = \{x\} \text{ for every } x \in p\}$ and $q < p$ if $p \subset q$. Has $\mathbb{P}$ got the countable chain condition?

Right separated sequences

For an Eberlein-Grothendieck right-separated transfinite sequence $X = \{x_\alpha : \alpha < \lambda\}$ we showed that X is hereditarily metalindelöf for $\lambda < \omega_2$. Moreover we proved that hereditarily metalindelöfness implies σ-discreteness of X for $\lambda < \omega_1 \cdot \omega_1$. It is not yet clear if hereditarily metalindelöfness implies σ-discreteness of X for $\lambda = \omega_1 \cdot \omega_1$.

- Suppose that $\omega_1 \cdot \omega_1 \leq \lambda < \omega_2$ and $X = \{x_\alpha : \alpha < \lambda\}$ is an Eberlein-Grothendieck right-separated transfinite sequence. Is X σ-discrete?

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