Countable Fréchet groups

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Kakutani-Birkhoff theorem

Theorem (Kakutani-Birkhoff)

A T_1 topological group is metrizable if and only if it is first countable.

Malykhin's problem

Definition

A topological space X is *Fréchet* if for every $A \subseteq X$ and every $x \in \bar{A}$ there is a seqence $\langle x_n : n \in \omega \rangle$ of elements of A converging to x.

Problem (Malykhin)

Is there a countable Fréchet group that is not metrizable?

Problem (Juhász)

Is there a countable Fréchet space of uncountable π-weight?

π-weight $\pi(X)$ of a space X is the minimal size of a *π-base*, i.e. a family of non-empty open sets such that every non-empty open set contains an element of the family.

Theorem (Archangel'skii)

Weight and π-weight coincide for topological groups.

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Partial positive solutions (to both problems):

- $\mathfrak{p} > \omega_1$ ... Yes
- (Gerlits-Nagy) There is an uncountable γ-set ... Yes
- (Nyikos) $\mathfrak{p} = \mathfrak{b}$ ... Yes
- (Ohrenstein-Tsaban) $\mathfrak{p} = \mathfrak{b}$ there is an uncountable γ-set.

Recall that a set of reals Y is a *γ-set* if every open ω-cover of Y has a γ-subcover. A cover $\mathcal{U}$ is an

- *ω-cover* if every finite subset of Y is contained in an element of $\mathcal{U}$,
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Barman-Dow Theorem

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It is consistent with **ZFC** that every countable Fréchet space has π-weight at most $\aleph_1$.

Corollary

It is consistent with **ZFC** that every separable Fréchet group has weight at most $\aleph_1$.

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On the other hand...

Theorem (H.-Ramos García)

It is relatively consistent with the continuum arbitrarily large that every countable Fréchet space of weight less than continuum has a countable π-base.

Corollary

It is relatively consistent with the continuum arbitrarily large that every countable (separable) Fréchet group of weight less than continuum is metrizable.

Question

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Theorem (Ol'shanskii)

There is a countable group admitting no non-discrete group topology.

Definition

A group is *topologizable* if it admits a non-discrete group topology.

Observation ($\mathfrak{p} > \omega_1$)

Every topologizable countable group admits a non-metrizable Fréchet group topology.

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Separable precompact groups

Theorem (Archangel'skii)

Every compact Fréchet group is metrizable.

Definition

A group is *precompact (totally bounded)* if it admits a group compactification (finitely many translates of every neighbourhood of the identity cover the group).

Theorem (H.-Ramos García) (There is an uncountable γ-set.)

Every countable abelian group admits a non-metrizable Fréchet group topology.

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The model

Theorem (H.-Ramos García)

There is a model of **ZFC** in which

1. the continuum is arbitrarily large,
2. every countable Fréchet space of weight less than continuum has a countable π-base,
3. every separable precompact Fréchet group is metrizable.

Plan of the proof

Using a standard bookkeeping arguments we construct a FS iteration of length κ (κ a sufficiently large regular cardinal) σ-centered forcing notions, eventually taking care of all countable Fréchet spaces of π-weight less than κ. At stage α when dealing with the space X_α handed to us by the bookkeeping device we need to do two things:

1. add a set $A \subseteq X_\alpha$ which has a point x as an accumulation point, and does not have a subsequence convergent to x, and
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Fréchet idealized

- Given a space X and a point $x \in X$ we denote by $\mathcal{I}_x$ the dual ideal to the filter of neighbourhoods of x, $\mathcal{I}_x = \{A \subseteq X : x \notin \bar{A}\}$.
- If X is countable then the infinite members of $\mathcal{I}^\perp = \{J \subseteq X : (\forall I \in \mathcal{I}) |I \cap J| < \omega\}$ are exactly the sequences convergent to x.
- The space X is Fréchet at x iff every $\mathcal{I}_x$-positive set contains an infinite element of $\mathcal{I}_x^\perp$ iff $\mathcal{I}_x^{\perp\perp} = \mathcal{I}_x$ iff for no $A \in \mathcal{I}_x^+$ is the ideal $\mathcal{I}_x \upharpoonright A$ tall.
- Call an ideal $\mathcal{I}$ on X *countably tall (ω-hitting)* if for every $\langle X_i : i \in \omega \rangle \subseteq X$ there is an $I \in \mathcal{I}$ such that $|X_i \cap I| = \omega$ for all $i \in \omega$.
- A forcing notion $\mathbb{P}$ *seals* an ideal $\mathcal{I}$ if it adds an $\mathcal{I}$-positive set A such that the ideal $\mathcal{I} \upharpoonright A$ is countably tall.

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Preservation of countable tallness

Definition

A forcing notion $\mathbb{P}$ *strongly preserves countable tallness* if for every sequence $\langle \dot{A}_n \colon n \in \omega \rangle$ of $\mathbb{P}$-names for infinite subsets of ω there is a $\langle B_n \colon n \in \omega \rangle$ sequence of infinite subsets of ω such that for any $B \in [\omega]^\omega$, if $B \cap B_n$ is infinite for all n then $\Vdash_{\mathbb{P}}$ "$B \cap \dot{A}_n$ is infinite for all n".

Proposition (Brendle-H.)

Finite support iteration of forcings strongly preserving countable tallness strongly preserves countable tallness.

So the plan is to:

1. seal, and
2. strongly preserve countable tallness.

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The Laver-Prikry forcing

Definition

Let $\mathcal{F}$ be a filter on ω. The *Laver-Prikry* forcing associated with $\mathcal{F}$ is the set of those trees $T \subseteq \omega^{<\omega}$ for which there is $s_T \in T$ (the *stem of* T) such that for all $s \in T$, $s \subseteq s_T$ or $s_T \subseteq s$ and such that for all $s \in T$, with $s \supseteq s_T$ the set $\text{succ}_T(s) = \{n \in \omega \colon s^\frown n \in T\} \in \mathcal{F}$, ordered by inclusion.

$\mathbb{L}_\mathcal{F}$ is a σ-centered forcing which adds generically a function $\dot{\ell}_\mathcal{F} \colon \omega \to \omega$. Denote by $\dot{A}_{gen}$ the canonical name for the range of $\dot{\ell}_\mathcal{F}$.

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Preservation of countable tallness by $\mathbb{L}_{\mathcal{F}}$

Lemma 1 (Brendle-H.)

Let $\mathcal{I}$ be an ideal on ω and let $\mathcal{F} = \mathcal{I}^*$ be the dual filter. Then the following are equivalent:

(1) For every $A \in \mathcal{I}^+$ and every $\mathcal{J} \leqslant_K \mathcal{I} \restriction A$ the ideal $\mathcal{J}$ is not countably tall.

(2) $\mathbb{L}_{\mathcal{F}}$ strongly preserves countable tallness.

(3) $\mathbb{L}_{\mathcal{F}}$ preserves countable tallness.

Recall the definition of the *Katětov order*: Given two ideals $\mathcal{I}$, $\mathcal{J}$ on ω, we say that $\mathcal{I} \leqslant_K \mathcal{J}$ if there is a function $f : \omega \to \omega$ such that $f^{-1}[I] \in \mathcal{J}$ for every $I \in \mathcal{I}$.

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Sealing $\mathcal{I}$ by $\mathbb{L}_{\mathcal{F}}$

Lemma 2 (H.-Ramos García)

Let $\mathcal{I}$ be an ideal on ω and let $\mathcal{F}$ be al filter on ω.

IF

$\mathcal{I} \cap \mathcal{F} = \emptyset$ and for every countable family $\mathcal{H} \subseteq \mathcal{F}^+$ there is an $I \in \mathcal{I}$ such that $H \cap I \in \mathcal{F}^+$ for all $H \in \mathcal{H}$ (i.e. $\mathcal{I}$ is ω-*hitting w.r.t.* $\mathcal{F}^+$)

THEN

the forcing $\mathbb{L}_{\mathcal{F}}$ seals the ideal $\mathcal{I}$.

Forcing with $\mathbb{L}_{\text{dense open}}$

Main Proposition

Let $X = (\omega, \tau)$ be a regular Fréchet space, $x \in X$ be such that $\pi\chi(x, X) > \omega$. Let $\mathcal{G}$ be the filter of dense open subsets of X. Then:

(1) $\mathbb{L}_{\mathcal{G}}$ seals $\mathcal{I}_x$, and

(2) $\mathbb{L}_{\mathcal{G}}$ strongly preserves countable tallness.

The precompact case

Lemma (H.-Ramos García)

Let $\mathbb{G} = \{g_n : n \in \omega\}$ be a countable group and $f : \omega \to \mathbb{G}$. Then the set

$$A_f = \bigcup_{n \in \omega} f(n) \cdot \{g_k : k < n\}$$

is dense in every precompact topology on $\mathbb{G}$

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Does Malykhin's problem have a negative solution in our model?

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Is it consistent that every countable Fréchet space has countable π-weight?

Theorem (Dow) ($\mathfrak{b} = \mathfrak{c}$)

There is a countable Fréchet space of π-weight $\mathfrak{c}$.

Corollary ($\mathfrak{c} \leq \omega_2$)

There is a countable Fréchet space of uncountable π-weight.

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Is it consistent that there is a non-metrizable countable Fréchet group but there are no uncountable γ-sets?

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