The operator ideal of weakly compactly generated operators

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Background

Theorem (B. E. Johnson 1967). Let X be a Banach space which is isomorphic to its square $X \oplus X$. Then each derivation from the Banach algebra $\mathscr{B}(X)$ of (bounded) operators on X into a Banach $\mathscr{B}(X)$-bimodule is automatically continuous.

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It is defined using a representation of operators on $C[0, \omega_1]$ as scalar-valued $[0, \omega_1] \times [0, \omega_1]$-matrices; an operator belongs to $\mathscr{M}$ if and only if its final column is continuous.

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Theorem (TK+Niels Laustsen, JFA 2012). An operator on $C[0,\omega_1]$ belongs to the Loy–Willis ideal if and only if the identity operator on $C[0,\omega_1]$ does not factor through it;

$$\mathscr{M} = \{T \in \mathscr{B}(C[0,\omega_1]) : \forall\, R, S \in \mathscr{B}(C[0,\omega_1])\colon I \neq STR\}.$$

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Corollary. The Loy–Willis ideal is the unique maximal ideal of $\mathscr{B}(C[0,\omega_1])$.

Proof. The theorem implies that the identity operator belongs to the ideal generated by any operator not in $\mathscr{M}$. $\square$

Related results

Many Banach spaces X share with $C[0, \omega_1]$ the property that

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Note that $C[0,\omega_1]$ differs from all of the above-mentioned Banach spaces because $C[0,\omega_1] \not\cong C[0,\omega_1] \oplus C[0,\omega_1]$.

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- containing the identity operator on the one-dimensional Banach space
- which assigns to each pair (E, F) of Banach spaces a (not necessarily closed) linear subspace $\mathscr{J}(E, F) = \mathscr{B}(E, F) \cap \mathscr{J}$ such that for any Banach spaces X, Y, E, F and for any operators $T \in \mathscr{B}(X, E), S \in \mathscr{J}(E, F)$ and $R \in \mathscr{B}(F, Y)$ we have $RST \in \mathscr{J}(X, Y)$.

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Examples: Compact operators $\mathscr{K}$, weakly comapct operators $\mathscr{W}$, strictly singular operators $\mathscr{S}$, operators with range of density character $< \kappa$ $\mathscr{X}_\kappa$ (denoted $\mathscr{X}_{\omega_1} = \mathscr{X}$), weak Banach–Saks operators $\mathscr{W}\mathscr{B}\mathscr{S}$ and many more...

Weakly compactly generated operators

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Theorem (P. Koszmider, N. Laustsen, TK; joint work in progress) For each $T \in \mathscr{B}(C_0([0, \omega_1)))$ there are a scalar λ and a club set $D \subseteq \omega_1$ such that

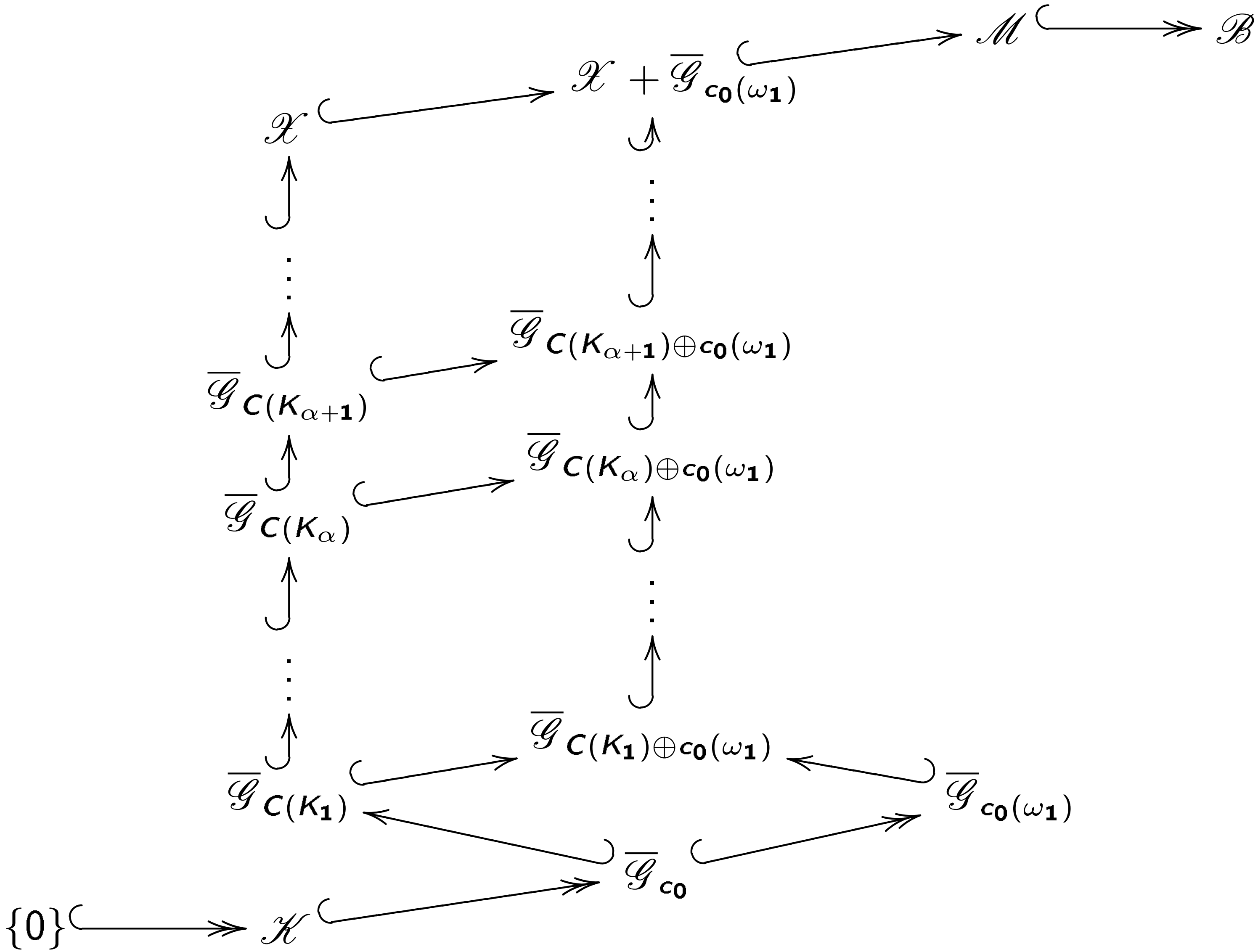
$$e_\alpha^* T(x) = \lambda e_\alpha^* x$$

for $x \in C_0([0, \omega_1))$, $\alpha \in D$ (e_α^* is the Dirac point mass δ_α functional).

Theorem (T.K., T. Kochanek). A similar theorem holds for the long James space $\mathscr{J}_p(\omega_1)$ (recall its norm)

$$\|x\|_{\mathscr{J}_p} = 2^{-1/p} \sup \Big\{ \Big(|x(\alpha_n) - x(\alpha_0)|^p + \textstyle\sum_{j=1}^n |x(\alpha_j) - x(\alpha_{j-1})|^p \Big)^{1/p} : n \in \mathbb{N} \text{ and } 0 \leqslant \alpha_0 < \alpha_1 < \ldots < \alpha_n < \eta \Big\},$$

Partial structure of the lattice of closed ideals of $\mathscr{B} = \mathscr{B}(C[0,\omega_1])$



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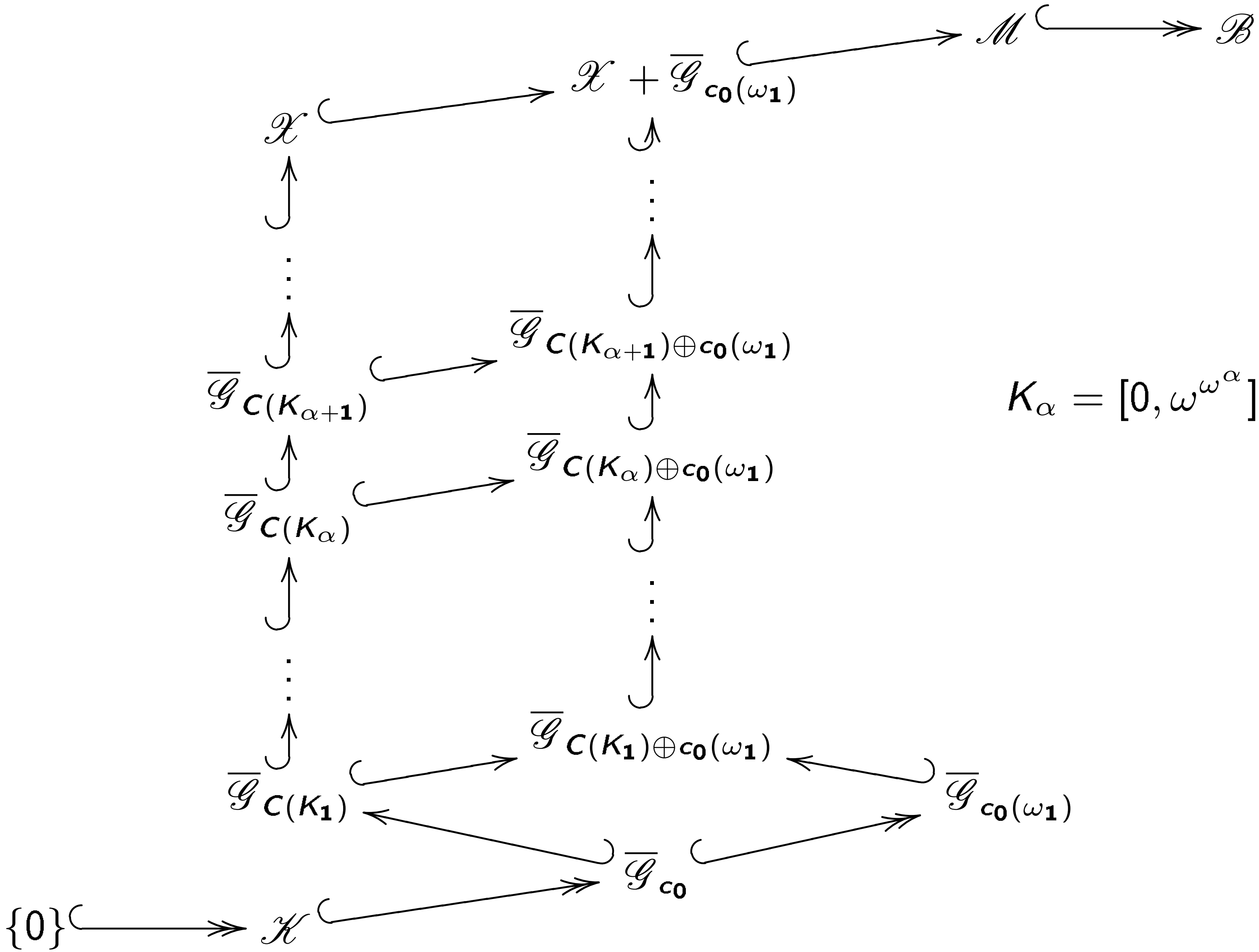

$K_\alpha = [0, \omega^{\omega^\alpha}]$

Weakly compactly generated operators

Corollary (P. Koszmider, N. Laustsen, TK) The map $\Lambda T \mapsto \lambda$ as above is a character on $\mathscr{B}(C_0(\omega_1))$ (and $\mathscr{B}(\mathscr{J}_p)$. For $T \in \mathscr{B}(C_0(\omega_1))$ we have $T \in \ker \Lambda \iff$ the range of T is contained in $C_0(\omega_1 \setminus D)$ which turns out to be a WCG Banach space.

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Question The above space K is an example of a Mrówka space, that is, the Stone space of a Boolean subalgebra of $\wp(\omega)$ generated by some uncountable almost disjoint family (and finite sets). Is for every Mrówka space K the ideal $\mathscr{WCG}(C(K))$ maximal ideal of $\mathscr{B}(C(K))$.