Universality properties of ℓ_∞/c_0

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A compact space K is *uniform Eberlein* if

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General Problem

Let $\mathcal{K}$ be some class of compact spaces of weight $\leqslant \mathfrak{c}$. Is it true that $C(K)$ embeds isomorphically (isometrically) into ℓ_∞/c_0 for any $K \in \mathcal{K}$?

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For any compact space K of weight $\leqslant \aleph_1$, $C(K)$ embeds isometrically into ℓ_∞/c_0.

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Theorem (Brech-Koszmider)

Consistently, there is a uniform Eberlein compactum K such that $C(K)$ does not embed isomorphically into ℓ_∞/c_0.

Definition

A cardinal κ is *Kunen* if each subset of $\kappa \times \kappa$ belongs to the σ-field generated by sets of the form $A \times B$, for $A, B \subseteq \kappa$.

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(iii) (Kunen) consistently $\mathfrak{c}$ is not a Kunen cardinal

Theorem (Todorčević 2011)

If $\mathfrak{c}$ is not a Kunen cardinal, then there is a Corson compactum K of weight $\leqslant \mathfrak{c}$ such that $C(K)$ does not embed isometrically into ℓ_∞ / c_0.

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Theorem

If $\mathfrak{c}$ is not a Kunen cardinal, then there is a uniform Eberlein compactum K of weight $\leqslant \mathfrak{c}$ such that $C(K)$ does not embed isometrically into ℓ_∞/c_0.

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WLOG, we may assume that $\forall a, b \in E \ \ a < b$.
Consider the space

$$K_2(E) = \{\chi_A \in \{0,1\}^{\mathbb{R}} : A \in [\mathbb{R}]^{\leqslant 2}, \forall a, b \in A \ \ a < b \Rightarrow (a, b) \in E\}.$$

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Now (by Todorčević 2011) we show that $C(K_2(E))$ does not embed isometrically into ℓ_∞ / c_0.

Theorem

If $\mathfrak{c}$ is not a Kunen cardinal then there exists a uniform Eberlein compact space K such that the space $C(K)$ embeds isomorphically, but fails to embed isometrically into ℓ_∞/c_0.

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If $\mathfrak{c}$ is not a Kunen cardinal then there exists a uniform Eberlein compact space K such that the space $C(K)$ embeds isomorphically, but fails to embed isometrically into ℓ_∞/c_0.

Proof

The space $K_2(E)$ considered in the proof of the previous theorem is a uniform Eberlein compactum such that $C(K_2(E))$ does not embed isometrically into ℓ_∞/c_0.

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If $\mathfrak{c}$ is not a Kunen cardinal then there exists a uniform Eberlein compact space K such that the space $C(K)$ embeds isomorphically, but fails to embed isometrically into ℓ_∞/c_0.

Proof

The space $K_2(E)$ considered in the proof of the previous theorem is a uniform Eberlein compactum such that $C(K_2(E))$ does not embed isometrically into ℓ_∞/c_0. However, it embeds isomorphically into ℓ_∞/c_0, since it is isomorphic to $c_0(\mathfrak{c})$ (Marciszewski 2003).

Problem (Benyamini, Rudin, Wage 1977)

Is it true that any uniform Eberlein compactum of weight $\leqslant \kappa$ is a continuous image of $A(\kappa)^\omega$, where $A(\kappa)$ denotes the one point compactification of a discrete space of size κ?

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For a binary relation $E \subseteq \mathbb{R}^2$ and integer $n \geqslant 2$, denote by $E^{[n]}$ the following set

$$E^{[n]} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : \forall i < j \ \ (x_i, x_j) \in E \ \text{ and } x_i \neq x_j\}$$

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Theorem (Todorčević 2011)

Suppose that for every Corson compact space K of weight $\leqslant \mathfrak{c}$, the space $C(K)$ embeds isomorphically into ℓ_∞ / c_0. Then for every binary relation $E \subseteq \mathbb{R}^2$ and for all but finitely many positive integers n, the set $E^{[n]}$ can be separated from $(E^c)^{[n]}$ by a member of $\mathcal{P}^n(\mathbb{R})$

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