Recent developments in the Fraïssé-Jónsson theory

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Outline

Fraïssé-Jónsson theory

Given a class $\mathfrak{K}$ of "small" objects, we are asking for a "large" object, reachable from $\mathfrak{K}$, that is universal for $\mathfrak{K}$ and homogeneous with respect to $\mathfrak{K}$-substructures.

- Fraïssé 1954; Jónsson 1960
- Droste & Göbel 1989: Category-theoretic approach
- Irwin & Solecki 2006: Reversed Fraïssé limits

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General assumptions

We fix a category $\mathfrak{K}$ of “small” objects, satisfying the following conditions:

1. $\mathfrak{K}$ has the Amalgamation Property.
2. $\mathfrak{K}$ has a weakly initial object 0, that is, $\mathfrak{K}(0, x) \neq \emptyset$ for every $\mathfrak{K}$-object x.

The Amalgamation Property:

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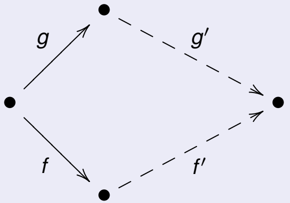

Fraïssé sequences

Crucial definition:

A sequence

$$u_0 \longrightarrow u_1 \longrightarrow u_2 \longrightarrow \cdots$$

is Fraïssé

if for every n, for every $\mathfrak{K}$-arrow $f\colon u_n \to y$ there exist $m \geqslant n$ and a $\mathfrak{K}$-arrow $g\colon y \to u_m$ such that $g \circ f = u_n^m$.

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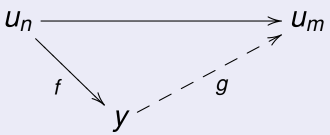

Existence

Theorem

Let $\kappa \geqslant \aleph_0$ be a regular cardinal. Assume $\mathfrak{K}$ is κ-bounded and dominated by $\leqslant \kappa$ arrows. Then $\mathfrak{K}$ has a Fraïssé sequence of length κ.

Definition

A category $\mathfrak{K}$ is κ-bounded if every sequence of length $< \kappa$ has an upper bound in $\mathfrak{K}$.

An upper bound for a sequence $\vec{x} \colon \delta \to \mathfrak{K}$ is a $\mathfrak{K}$-object y and a collection of $\mathfrak{K}$-arrows $f_\xi \colon x_\xi \to y$ such that

$$f_\xi = f_\eta \circ x_\xi^\eta$$

whenever $\xi < \eta < \varrho$.

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Universality

Theorem

Let $\vec{u}$ be a Fraïssé sequence in $\mathfrak{K}$. Let $\vec{x}$ be a continuous sequence of length $\leqslant \text{length}(\vec{u})$. Then there exists an arrow

$$\vec{f} \colon \vec{x} \to \vec{u}$$

in the category of $\mathfrak{K}$-sequences.

Definition

A sequence $\vec{x}$ is continuous if for every limit ordinal $\delta < \text{length}(\vec{x})$, it holds that $x_\delta = \lim(x \restriction \delta)$.

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Theorem (Uniqueness)

Let $\vec{u}$ and $\vec{v}$ be continuous Fraïssé sequences of the same regular length. Then

$$\vec{u} \approx \vec{v}$$

in the category of sequences.

Theorem (Homogeneity)

Let $\vec{u}$ be a continuous Fraïssé sequence and let $i \colon a \to \vec{u}$, $j \colon b \to \vec{u}$ be such that a, b are $\mathfrak{K}$-objects. Then for every isomorphism $h \colon a \to b$ there exists an automorphism $H \colon \vec{u} \to \vec{u}$ for which the diagram

$$\begin{array}{ccc} \vec{u} & \xrightarrow{H} & \vec{u} \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle j} \\ a & \xrightarrow[h]{} & b \end{array}$$

is commutative.

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A counterexample

Theorem

There exists a category of countable binary trees with uncountably many pairwise incomparable Fraïssé sequences of length ω_1.

- Objects: Countable complete binary trees
- Arrows: Embeddings onto initial segments

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Embedding-Projection Pairs

Definition (cf. Droste & Göbel 1989)

Fix a category $\mathfrak{K}$. The category $\ddagger\mathfrak{K}$ of embedding-projection pairs is defined as follows:

- The objects of $\ddagger\mathfrak{K}$ are the objects of $\mathfrak{K}$.
- An arrow from a to b is a pair $\langle e, r \rangle$, where $e \colon a \to b$, $r \colon b \to a$ are $\mathfrak{K}$-arrows satisfying

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The Cantor Set

Theorem

There exists a continuous function $u\colon 2^\omega \to 2^\omega$ with the following property:

- *Given a continuous map $f\colon K \to L$ between 0-dimensional compact metric spaces, there exist embeddings $i\colon K \to 2^\omega$, $j\colon L \to 2^\omega$ and retractions $r\colon 2^\omega \to K$, $s\colon 2^\omega \to L$ such that the diagrams*

$$\begin{array}{ccc} 2^\omega & \xrightarrow{u} & 2^\omega \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle j} \\ K & \xrightarrow[f]{} & L \end{array} \qquad\qquad \begin{array}{ccc} 2^\omega & \xrightarrow{u} & 2^\omega \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle s} \\ K & \xrightarrow[f]{} & L \end{array}$$

  commute.

Theorem

There exists a sequence of continuous maps $\{u_n \colon 2^\omega \to 2^\omega\}_{n\in\omega}$ *with the following property:*

- *Given a sequence of continuous maps* $\{f_n \colon K \to L\}_{n\in\omega}$ *between 0-dimensional compact metric spaces, there exist embeddings* $i \colon K \to 2^\omega$, $j \colon L \to 2^\omega$, *retractions* $r \colon 2^\omega \to K$, $s \colon 2^\omega \to L$, *and a strictly increasing function* $\varphi \colon \omega \to \omega$, *such that for each* $n \in \omega$ *the diagrams*

$$
\begin{array}{ccc}
2^\omega & \xrightarrow{u_{\varphi(n)}} & 2^\omega \\
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are commutative.

Banach spaces

Theorem

Assume CH. There exists a Banach space V of density $\aleph_1$ and with the following properties:

1. *V contains isometric copies of all Banach spaces of density $\leqslant \aleph_1$.*
2. *Every linear isometry between separable subspaces of V extends to an auto-isometry of V.*

Theorem (Brech & Koszmider 2011)

It is consistent with ZFC that there is no isomorphically universal Banach space for density $\mathfrak{c}$.

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If CH holds then there exists a complementably universal Banach space for Schauder bases of length ω_1.

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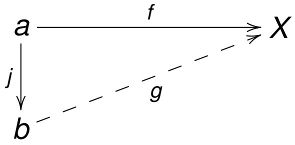
Retracts of Fraïssé limits

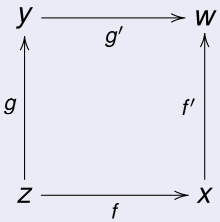
Injectivity

Definition

Let $\mathfrak{K} \subseteq \mathfrak{L}$ be two categories. An $\mathfrak{L}$-object X is $\langle \mathfrak{K}, \mathfrak{L} \rangle$-injective if for every $\mathfrak{K}$-arrow $j \colon a \to b$, for every $\mathfrak{L}$-arrow $f \colon a \to X$ there is an $\mathfrak{L}$-arrow $g \colon b \to X$ such that $g \restriction a = f$.



A pushout square



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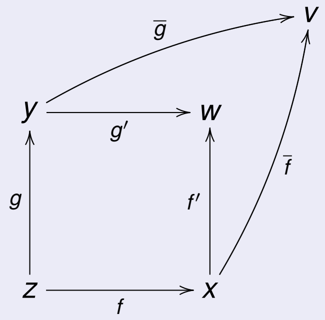

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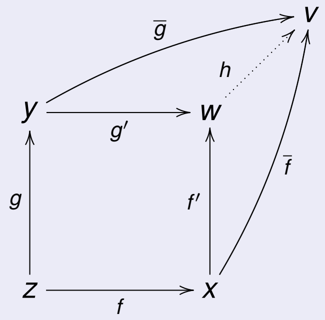

Definition

A pair of categories $\mathfrak{K} \subseteq \mathfrak{L}$ is nice if for every $\mathfrak{K}$-arrow $i\colon c \to a$, for every $\mathfrak{L}$-arrow $f\colon c \to b$, there exist an $\mathfrak{L}$-arrow $g\colon a \to w$ and a $\mathfrak{K}$-arrow $j\colon b \to w$ for which the diagram

$$\begin{array}{ccc} b & \xrightarrow{\;j\;} & w \\ {\scriptstyle f}\uparrow & & \uparrow{\scriptstyle g} \\ c & \xrightarrow[\;i\;]{} & a \end{array}$$

is a pushout square in $\mathfrak{L}$.

Theorem

Let $\mathfrak{K} \subseteq \mathfrak{L}$ be a nice pair of categories and let $\vec{u}$ be a Fraïssé sequence in $\mathfrak{K}$. For a $\mathfrak{K}$-sequence $\vec{x}$, the following properties are equivalent:

1. *$\vec{x}$ is a retract of $\vec{u}$.*
2. *$\vec{x}$ is $\langle \mathfrak{K}, \mathfrak{L} \rangle$-injective.*

Special cases: Dolinka 2011

Corollary

Let X be a Polish metric space. Then X is a non-expansive retract of the Urysohn space $\mathbb{U}$ if and only if X is finitely hyperconvex.

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Definition (Cameron & Nešetřil 2006)

A structure X is homomorphism homogeneous with respect to its "small" substructures if every homomorphism between "small" substructures of X extends to an endomorphism of X.

Theorem

Let $\mathscr{F}$ be a nice Fraïssé class with $U = \mathrm{Flim}(\mathscr{F})$. Given a countable structure in $\overline{\mathscr{F}}$, the following properties are equivalent:

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Metric categories

Motivation:

Theorem (Gurariĭ 1966)

There exists a separable Banach space $\mathbb{G}$ satisfying the following condition.

- *Given finite dimensional spaces $E \subseteq F$, given an isometric embedding $f\colon E \to \mathbb{G}$, for every $\varepsilon > 0$ there exists an extension $g\colon F \to \mathbb{G}$ of f such that $\|g\| \cdot \|g^{-1}\| < 1 + \varepsilon$.*

Theorem (Lusky 1976)

The Gurariĭ space is unique up to a linear isometry.

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Lemma (Solecki & K. 2011)

Let $f: X \to Y$ be an ε-isometric embedding of finite dimensional Banach spaces. Then there exist a finite dimensional Banach space Z and isometric embeddings $i: X \to Z$, $j: Y \to Z$ such that

$$\|j \circ f - i\| < \varepsilon.$$

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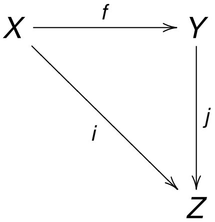


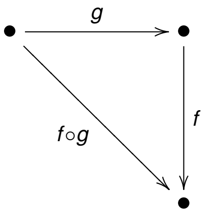
Metric categories

A *metric* on a category $\mathfrak{K}$ is a function $\mu\colon \mathfrak{K} \to [0, +\infty]$ satisfying the following conditions:

(M$_1$) $\mu(\mathrm{id}_x) = 0$ for every object x.

(M$_2$) $\mu(f \circ g) \leqslant \mu(f) + \mu(g)$.

(M$_3$) $\mu(g) \leqslant \mu(f \circ g) + \mu(f)$.



We further assume that $\mathfrak{K}$ is enriched over metric spaces.
That is, for each $\mathfrak{K}$-objects a, b a metric ϱ is defined on $\mathfrak{K}(a, b)$ so that

(M$_4$) $\varrho(f \circ h, g \circ h) \leqslant \varrho(f, g)$

(M$_5$) $\varrho(k \circ f, k \circ g) \leqslant \varrho(f, g)$

Moreover, the compatibility of μ and ϱ says:

(M$_6$) μ is uniformly continuous with respect to ϱ.

Prototype example

Let $\mathfrak{K}$ be the category of metric spaces with non-expansive maps and define
$$\mu(f) = \log \operatorname{Lip}(f^{-1}).$$

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The Law of Return

Given $\varepsilon > 0$, there is $\eta > 0$, such that whenever f is a $\mathfrak{K}$-arrow with $\mu(f) < \eta$, then there exist $\mathfrak{K}$-arrows g, h with $\mu(g)$ and $\mu(h)$ arbitrarily small and

$$\varrho(g \circ f, h) < \varepsilon$$

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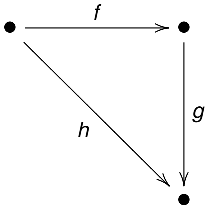

Definition

A sequence $\vec{x}$ is Cauchy if

$$(\forall\, \varepsilon > 0)(\exists\, n_0)(\forall\, m \geqslant n \geqslant n_0)\ \ \mu(x_n^m) < \varepsilon.$$

Denote by $\sigma\mathfrak{K}$ the category of all Cauchy sequences in $\mathfrak{K}$.

Claim

The functions μ and ϱ naturally extend from $\mathfrak{K}$ to $\sigma\mathfrak{K}$.

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The functions μ and ϱ naturally extend from $\mathfrak{K}$ to $\sigma\mathfrak{K}$.

Definition

A Cauchy sequence $\vec{u}\colon \omega \to \mathfrak{K}$ is Fraïssé if

- Given $\varepsilon > 0$, there are $\eta > 0$ and n_0 such that whenever $n \geqslant n_0$ and $f\colon u_n \to y$ is a $\mathfrak{K}$-arrow satisfying $\mu(f) < \eta$, there exist $m > n$ and a $\mathfrak{K}$-arrow $g\colon y \to u_m$ such that $\mu(g)$ is arbitrarily small and

$$\varrho(g \circ f, u_n^m) < \varepsilon.$$

Theorem

Assume $\langle \mathfrak{K}, \mu, \varrho \rangle$ is dominated by countably many arrows. Then there exists a Fraïssé sequence in $\mathfrak{K}$.

Theorem

Assume $\langle \mathfrak{K}, \mu, \varrho \rangle$ satisfies the Law of Return and let $\vec{u}$ be a Fraïssé sequence in $\mathfrak{K}$. Then:

1. *For every Cauchy sequence $\vec{x}$ there exists an arrow $F\colon \vec{x} \to \vec{u}$ such that $\mu(F) = 0$.*
2. *For every other Fraïssé sequence $\vec{v}$ there exists an isomorphism $H\colon \vec{u} \to \vec{v}$ such that $\mu(H) = 0$.*

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2. *For every other Fraïssé sequence $\vec{v}$ there exists an isomorphism $H \colon \vec{u} \to \vec{v}$ such that $\mu(H) = 0$.*

Theorem

Assume $\langle \mathfrak{K}, \mu, \varrho \rangle$ is dominated by countably many arrows. Then there exists a Fraïssé sequence in $\mathfrak{K}$.

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An application

Theorem (Garbulińska & K. 2012)

There exists a linear operator $u_\infty \colon \mathbb{G} \to \mathbb{G}$ *with* $\|u_\infty\| = 1$ *and with the following property:*

- *Given a linear operator* $T \colon X \to Y$ *between separable Banach spaces with* $\|T\| \leqslant 1$, *there exist isometric embeddings* $i \colon X \to \mathbb{G}$ *and* $j \colon Y \to \mathbb{G}$ *for which the following diagram commutes.*

$$\begin{array}{ccc} \mathbb{G} & \xrightarrow{u_\infty} & \mathbb{G} \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle j} \\ X & \xrightarrow[T]{} & Y \end{array}$$

Uncountable Fraïssé classes
(joint with Antonio Avilés)

A natural question

Assume $\mathscr{F}$ is an uncountable class of finite models with the pushout property. Does there exist an $\mathscr{F}$-universal and $\mathscr{F}$-homogeneous structure?

If so, could it be a Fraïssé-Jónsson limit?

Motivation:

- A. AVILÉS, C. BRECH, *A Boolean algebra and a Banach space obtained by push-out iteration*, Topology Appl. **158** (2011) 1534–1550

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Definition

Assume $\mathscr{F} \subseteq \mathfrak{C}$ and $\mathfrak{C}$ is a category with the pushout property. A $\mathfrak{C}$-arrow $f\colon x \to y$ is called an $\mathscr{F}$-cell if there are $\mathfrak{C}$-arrows $i\colon r \to x$, $j\colon s \to y$ and an $\mathscr{F}$-arrow $g\colon r \to s$ for which the square

$$\begin{array}{ccc} x & \xrightarrow{f} & y \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle j} \\ r & \xrightarrow[g]{} & s \end{array}$$

is a pushout in $\mathfrak{C}$.

Definition

An $\mathscr{F}$-cell complex is a continuous sequence $\vec{x} \colon \delta \to \mathfrak{C}$ such that x_0 is an object of $\mathscr{F}$ and $x_\alpha^{\alpha+1}$ is an $\mathscr{F}$-cell for every $\alpha < \delta$.

Source:

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Definition

Denote by $\mathfrak{K}_\delta(\mathscr{F})$ the subcategory of $\mathfrak{C}$ whose arrows are $\mathscr{F}$-cell complexes of length δ. Write

$$\mathfrak{K}_{<\kappa}(\mathscr{F}) = \bigcup_{\delta<\kappa} \mathfrak{K}_\delta(\mathscr{F}).$$

Theorem (Avilés & K.)

Assume κ is an infinite regular cardinal and $\mathfrak{C}$ is κ-continuous. Then the category $\mathfrak{K}_{<\kappa}(\mathscr{F})$ has a Fraïssé sequence of length κ.

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Theorem (Avilés & K.)

Assume $\kappa \geqslant |\mathscr{F}|$ and $\mathfrak{C} \supseteq \mathscr{F}$ is κ-continuous. There exists a unique $\mathfrak{K}_\kappa(\mathscr{F})$-object U which is $\mathfrak{K}_{<\kappa}(\mathscr{F})$-homogeneous and $\mathfrak{K}_\kappa(\mathscr{F})$-universal. In particular, U is $\mathscr{F}$-homogeneous.

Example

- $\mathfrak{C}$ = Boolean algebras with monomorphisms.
- $\mathscr{F}$ = finite Boolean algebras.

Claim

The objects of $\mathfrak{K}_\kappa(\mathscr{F})$ are projective Boolean algebras of size $\leqslant \kappa$.

Theorem (Shchepin 1976)

Let λ be an infinite cardinal. The free Boolean algebra with λ generators is the unique homogeneous projective Boolean algebra of cardinality λ.

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THE END