On the Banach-Saks property and convex hulls

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This is a joint work with C. Ruiz-Bermejo and P. Tradacete (Madrid)

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A classical theorem of Mazur asserts that the convex hull of a compact set in a Banach space is again relatively compact.

Indeed, Krein-Šmulian's Theorem states that the same holds even for weakly compact sets.

There is a well-known property lying between these two main kinds of compactness:

Definition

A subset $A \subseteq X$ of a Banach space is called Banach-Saks if every sequence in A has a Cesàro convergent subsequence, i.e. every sequence $(x_n)_n$ in A has a subsequence $(x_{n_k})_k$ such that the sequence of means

$$\left(\frac{1}{k}\sum_{i=1}^{k} x_{n_i}\right)_k$$

converges in norm.

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A space has the Banach-Saks property when its unit ball is a Banach-Saks set.

Examples of Banach-Saks sets are

1. the unit balls of ℓ_p's, $1 < p < \infty$
2. the unit basis of c_0.

Typical example of a weakly-null sequence which is not a Banach-Saks set is the unit basis of the Shreier space.

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Question

Is the convex hull of a Banach-Saks set again Banach-Saks?

By Ramsey-like methods, we show that the answer is No:

Theorem (LA-Ruiz-Tradacete)

There is a family $\mathcal{F}$ of finite subsets of $\mathbb{N}$ such that the unit basis of a Shreier-like space $X_{\mathcal{F}}$ is a Banach-Saks set, but its convex hull is not.

On the opposite direction we prove that

Theorem (LA-Ruiz-Tradacete)

Suppose that $\mathcal{F}$ is a "classical" family (i.e. if $\mathcal{F}$ is a generalized Schreier family), then the convex hull of a Banach-Saks subset of $X_{\mathcal{F}}$ is also Banach-Saks.

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Definition

Let $\mathcal{F}$ be a family on $\mathbb{N}$, i.e. a collection of (finite) subsets of $\mathbb{N}$. Given $x \in c_{00}(\mathbb{N})$ we define

$$\|x\|_{\mathcal{F}} := \max\{\|x\|_\infty, \sup_{s \in \mathcal{F}} \sum_{k \in s} |(x)_k|\}.$$

The Shreier-like space $X_{\mathcal{F}}$ is the completion of $(c_{00}(\mathbb{N}), \|\cdot\|_{\mathcal{F}})$.

It is easy to see that the unit basis $(u_n)_n$ of $c_{00}(\mathbb{N})$ is a 1-unconditional Schauder basis of $X_{\mathcal{F}}$.
The non-trivial spaces are coming from pre-compact families, i.e. such that $\overline{\mathcal{F}} \subseteq \mathrm{FIN}$.

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Let $\mathcal{S} = \{s \in \mathrm{FIN} \,:\, |s| \leq \min s + 1\}$ be the Schreier family.
Then the unit basis $\{u_n\}_n$ of $X_{\mathcal{S}}$ is a non-Banach-Saks:The main reason is that $\mathcal{S}$ is *large* in $\mathbb{N}$: For every $n \in \mathbb{N}$ and $M \subseteq \mathbb{N}$ there is $s \in \mathcal{S} \cap [M]^n$.
However,

Proposition (M. González, and J. Gutiérrez)

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The convex hull of a Banach-Saks subset of $X_\mathcal{S}$ is a Banach-Saks set.

Given two families $\mathcal{F}$ and $\mathcal{G}$ on $\mathbb{N}$, let

$$\mathcal{F} \oplus \mathcal{G} := \{s \cup t \,:\, s \in \mathcal{G},\, t \in \mathcal{F} \text{ and } s < t\}$$

$$\mathcal{F} \otimes \mathcal{G} := \{\bigcup_i s_i \,:\, \{s_i\}_i \subseteq \mathcal{F},\, s_i < s_j \text{ for } i < j, \text{ and } \{\min s_i\}_i \in \mathcal{G}\}.$$

Definition

For each $\alpha < \omega_1$, α limit, we fix a strictly increasing sequence $(\beta_n^{(\alpha)})_n$ such that $\sup_n \beta_n^{(\alpha)} = \alpha$. We define now

(a) $\mathcal{S}_0 := [\mathbb{N}]^{\leq 1}$.

(b) $\mathcal{S}_{\alpha+1} = \mathcal{S}_\alpha \otimes \mathcal{S}$.

(c) $\mathcal{S}_\alpha := \bigcup_{n\in\mathbb{N}} \mathcal{S}_{\beta_n^{(\alpha)}} \upharpoonright \mathbb{N}/n$.

Then each S_α is a compact, hereditary and spreading family.

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Theorem

For every pre-compact family $\mathcal{F}$ on $\mathbb{N}$ there are $\alpha < \omega_1$, $n \in \mathbb{N}$ and $M \subseteq \mathbb{N}$ such that

$$\mathcal{S}_\alpha[M] \oplus [M]^{\leq n} \subseteq \mathcal{F}[M] \subseteq \mathcal{S}_\alpha[M] \oplus [M]^{\leq n+1},$$

where $\mathcal{F}[M] := \{s \cap M \ : \ s \in \mathcal{F}\}$.

Theorem (LA-Ruiz-Tradacete)

The convex hull of a Banach-Saks subset of $X_{\mathcal{S}_\alpha}$ is also a Banach-Saks set.

Some definitions.

Definition

1. By a *family on an infinite subset* $M \subseteq \mathbb{N}$ we mean a collection of finite subsets of M.
2. A family $\mathcal{F}$ on M is called *large in* $N \subseteq M$ when for every infinite subset $P \subseteq N$ and every $n \in \mathbb{N}$ there is some $s \in \mathcal{F}$ such that $|s \cap P| \geq n$.
3. Given a partition $(I_n)_n$ of $\mathbb{N}$, *a transversal* (w.r.t. $(I_n)_n$) is a subset T of $\mathbb{N}$ such that $|T \cap I_n| \leq 1$ for every $n \in \mathbb{N}$.

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Definition

(a) There is some $\varepsilon > 0$ and some $M \subseteq \mathbb{N}$ such that the set

$$\mathcal{G}(\mathcal{F}, \varepsilon) := \{ t \subseteq \mathbb{N} : \exists s \in \mathcal{F} \, \forall n \in t \, \mu_n(s \cap I_n) \geq \varepsilon \}$$

is large in M.

(b) $\mathcal{F}[T] := \{ s \cap T : s \in \mathcal{F} \}$ is not large in T for every transversal $T \subseteq I$.

Definition

A *T-family* is a family $\mathcal{F}$ of finite subsets of $\mathbb{N}$ such that there is a partition $\bigcup_n I_n$ of $\mathbb{N}$ in finite pieces I_n and for each n probability measures μ_n on $\mathcal{P}(I_n)$ (i.e. a convex combination $(\lambda_k^{(n)})_{k\in I_n}$) with the following properties:

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Theorem (LA-Ruiz-Tradacete)

TFAE:

1. *The convex hull of every weakly-null Banach-Saks set is Banach-Saks.*
2. *There are no T-families.*

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There is a T-family (where indeed the measures μ_n on I_n are the counting measures on I_n).

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Proposition

Suppose that $\mathcal{F}$ is a T-family on $\mathbb{N}$ with respect to $(I_n)_n$ and $(\mu_n)_n$. Then

1. *Every subsequence of the unit basis $\bar{u} = (u_n)_n$ of $X_{\mathcal{F}}$ has a further subsequence equivalent to the unit basis of c_0; hence $\{u_n\}_n$ is a Banach-Saks set.*
2. *the set $\{\mu_n * \bar{x}\}_n \subseteq \mathrm{conv}(\{u_n\}_n)$ is not Banach-Saks.*

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Recall the following well-known fact:

Theorem (Gillis)

For every $\varepsilon > 0$, every $\delta > 0$ and every $m \in \mathbb{N}$ there is $n := n(\varepsilon, \delta, m)$ such that for every probability space $(\Omega, \mathcal{F}, \mu)$ and every sequence $(A_i)_{i<n}$ such that $\mu(A_i) \geq \varepsilon$ for all $i < n$, there is $s \in [n]^m$ such that

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Question

Given $\varepsilon > 0$ and $m \in \mathbb{N}$, does there exist $n := n(\varepsilon, m)$ such that whenever $(\Omega, \mathcal{F}, \mu)$ is a probability space and $(A_{\{i,j\}})_{\{i,j\} \in [n]^2}$ are such that $\mu(A_{\{i,j\}}) > \varepsilon$ for all $\{i,j\} \in [n]^2$, then there is $s \in [n]^m$ such that $\bigcap_{\{i,j\} \in [s]^2} A_{\{i,j\}} \neq \emptyset$?

NO: Example by *Erdös and Hajnal*: Fix arbitrary $n, r > 0$. For each $i, j \in [n]^2$, let

$$A_{\{i,j\}} := \{(a_k)_{k<n} \in r^n \, : \, a_i \neq a_j\}.$$

Consider r^n with its counting probability measure, $\mu(s) = |s|/r^n$ for $s \subseteq r^n$. Then

1. $\mu(A_{\{i,j\}}) \geq 1 - 1/r$, and
2. $\bigcap_{\{i,j\} \in [s]^2} A_{\{i,j\}} = \emptyset$ for every $s \subseteq n$ with $|s| \geq r + 1$.

Question

Given $\varepsilon > 0$ and $m \in \mathbb{N}$, does there exist $n := n(\varepsilon, m)$ such that whenever $(\Omega, \mathcal{F}, \mu)$ is a probability space and $(A_{\{i,j\}})_{\{i,j\}\in[n]^2}$ are such that $\mu(A_{\{i,j\}}) > \varepsilon$ for all $\{i,j\} \in [n]^2$, then there is $s \in [n]^m$ such that $\bigcap_{\{i,j\}\in[s]^2} A_{\{i,j\}} \neq \emptyset$?

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