Spaces with few operators

and the lack of complex structure

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References

P. Koszmider, M. Martín, and J. Merí
Extremely non-complex $C(K)$ spaces
J.Math. Anal. Appl., 350 (2009) 601–615

P. Koszmider, M. Martín, and J. Merí
Isometries on extremely non-complex Banach spaces
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Notation

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X Banach space

- X^* is the topological dual of X
- $L(X)$ denotes the space of all (bounded linear) operators on X
- $W(X)$ is the subspace of $L(X)$ containing all weakly compact operators on X
- Id denotes the identity operator

Complex structure

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- Banach spaces such that every operator is of the form $\lambda \mathrm{Id} + S$ with S strictly singular

The problem

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- X cannot have unconditional basis (Martín-Merí-2011)

The first examples: $C(K)$ spaces with few operators

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Theorem (weak multiplications)

Let K be a perfect compact space so that every operator on $C(K)$ is of the form $g\mathrm{Id} + S$ where $g \in C(K)$ and $S \in W(C(K))$. Then, $C(K)$ is extremely non-complex.

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Let K be a perfect compact space so that for every operator $T \in L(C(K))$ one has $T^* = g\mathrm{Id} + S$ where g is a bounded Borel function on K and $S \in W(C(K)^*)$. Then, $C(K)$ is extremely non-complex.

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Consequence

There are infinitely many nonisomorphic extremely non-complex Banach spaces.

- If $C(K)$ is extremely non-complex and $|K| \geqslant 2$ then K is perfect.

Other examples: spaces failing to have few operators

Theorem

There are compact spaces K_1, K_2 such that $C(K_1)$ and $C(K_2)$ are extremely non-complex and

- $C(K_1)$ contains a complemented copy of $C(2^\omega)$
- $C(K_2)$ contains a (complemented) copy of ℓ_∞

Further examples: large subspaces of $C(K)$ spaces

- K compact space
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Theorem

Let K be a perfect compact space so that every operator on $C(K)$ is a weak multiplier. Then, $C_E(K\|L)$ is extremely non-complex.

Producing Banach spaces with trivial group of onto isometries

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Theorem

Let K be a connected compact space so that $K \setminus L$ is also connected and $C_E(K\|L)$ is extremely non-complex. Then $\mathrm{Iso}(C_E(K\|L)) = \{\mathrm{Id}, -\mathrm{Id}\}$.

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Theorem

There exist K and L satisfying:

- K is connected
- $K \setminus L$ is connected
- All operators on $C(K)$ are weak multipliers ($C_E(K\|L)$ is extremely non-complex)
- L maps continuously onto the Cantor set (any separable E embeds in $C(L)$)

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Take K and L as above and $E = \ell_2$. Then:

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Take K and L as above and $E = \ell_2$. Then:

- $\mathrm{Iso}(C_E(K\|L)) = \{\mathrm{Id}, -\mathrm{Id}\}$
- $\mathrm{Iso}(C_E(K\|L)^*) \supset \mathrm{Iso}(\ell_2^*)$ (recall that $C_E(K\|L)^* \equiv C_0(K\|L)^* \oplus_1 E^*$)

A couple of questions

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- Is there a hereditarily indecomposable extremely non-complex Banach space?

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