Separating the bounding and dominating numbers for classes of uncountable structures

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Trends in Set Theory, Warszawa
10 July 2012

Setting

Let $\kappa \leqslant \lambda$ always denote uncountable regular cardinals with $\lambda^{<\lambda} = \lambda$.

Let us consider a class of structures, each of size λ, which do not have specific substructures of size κ, such as

- trees of size λ without branches of length κ,
- posets of size λ without increasing κ-chains or chains of size κ,
- graphs of size λ without cliques of size κ.

We consider the quasi-order on the class defined by embeddings or (for trees) strict order-preserving maps.

The original motivation comes from subtrees of $\lambda^{<\lambda}$ without cofinal branches. Universal families of such trees are useful for studying Π^1_1 subsets of ${}^{\lambda}\lambda$.

Universal families

A structure $M \in \mathcal{C}$ is universal for a class $\mathcal{C}$ if every $N \in \mathcal{C}$ embeds into M.

A family $F \subseteq \mathcal{C}$ is universal for $\mathcal{C}$ if every $N \in \mathcal{C}$ embeds into some $M \in F$.

Fact

Suppose Φ is a countable complete first-order theory (and $\lambda^{<\lambda} = \lambda$). Then Φ has a λ-saturated and hence universal model of size λ.

The classes which we study are not first-order axiomatizable.

Trees

Let T_λ denote the class of trees of height and width $\leqslant \lambda$ without cofinal branches.

Fact (Todorcevic)

Let σT denote the set of linearly ordered downwards closed subsets of T ordered by extension, for $T \in T_\lambda$. Then there is no strict order-preserving map $\sigma T \to T$. Hence there is no universal tree in T_λ (assuming $\lambda^{<\lambda} = \lambda$).

Bounding and dominating numbers

Definition

The bounding number $b = b(\mathcal{C})$ is the least size of an unbounded family in $\mathcal{C}$.

The dominating number or universality number $d = d(\mathcal{C})$ is the least size of a universal family in $\mathcal{C}$.

Note that $b = cf(b) \leqslant cf(d) \leqslant d$.

When λ structures can be glued together to a single structure in the class, we have $b(\mathcal{C}) \geqslant \lambda^+$.

The Cohen model

Adding $\mu > \lambda^+$ Cohen subsets of λ over a model of $2^\lambda = \lambda^+$

- does not add trees in T_λ dominating T_λ^V and
- adds trees that are not dominated by any tree in T_λ^V.

So $b(T_\lambda) = \lambda^+$ and $d(T_\lambda) = 2^\lambda$ in the extension.

Previous results

The bounding and dominating number for a class can be changed simultaneously by iteratively adding dominating structures.

Theorem (Mekler-Väänänen)

For any regular μ with $\lambda^+ \leqslant \mu \leqslant 2^\lambda$, there is a $< \lambda$-closed λ^+-c.c. extension with $b(T_\lambda) = d(T_\lambda) = \mu$.

Theorem (Komjath-Shelah)

Let $\mathcal{C}$ denote the class of graphs of size λ without cliques of size κ. There is a λ-strategically closed λ^+-c.c. extension with $d(\mathcal{C}) = \lambda^+$ and 2^λ large.

Previous results continued

Notice that the bounding and dominating numbers cannot be separated for cardinals of countable cofinality.

Theorem (Dzamonja-Väänänen)

If $cf(\mu) = \omega$, *then* $b(T_\mu) = d(T_\mu) = \mu^+$.

Theorem (Hajnal-Komjath)

For the class $\mathcal{C}$ *of graphs of size* λ *without cliques of size* ω, $d(\mathcal{C}) = \lambda^+$.

Results

Suppose that Q is a poset with $b(Q) \geqslant \lambda^+$ (and $\lambda^{<\lambda} = \lambda$).

Theorem

There is a $< \lambda$-closed λ^+-c.c. extension with $b(T_\lambda) = b(Q)$ and $d(T_\lambda) = d(Q)$.

Theorem

Let $\mathcal{C}$ denote the class of posets of size λ without increasing λ-chains. There is a $< \lambda$-closed λ^+-c.c. extension with $b(\mathcal{C}) = b(Q)$ and $d(\mathcal{C}) = d(Q)$.

Proposition

There is a $< \lambda$-closed λ^+-c.c. forcing adding a dominating poset of size λ without chains of size λ.

Nonlinear iterations

Definition

Let Q be a wellfounded poset and $Q \downarrow a = \{q \in Q \mid q < a\}$. A nonlinear iteration along Q with $< \lambda$-support is a sequence $(P_a, \dot{P}_a : a \in Q)$ such that for $p \in P_a$

- $supt(p) \subseteq Q \downarrow a$, $|supt(p)| < \lambda$,
- *$p(b)$ is a P_b-name for a condition in $\dot{P}_b$ for all $b \in Q \downarrow a$,*

and $p \leqslant q$ if

- $supt(q) \subseteq supt(p)$,
- *$p \restriction (Q \downarrow a) \Vdash_{P_a} p(a) \leqslant_{\dot{P}_a} q(a)$ for all $a \in supt(q)$.*

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Hechler used a nonlinear iteration to change the bounding and dominating numbers of $({}^{\omega}\omega, <^*)$. Cummings and Shelah generalized this to $({}^{\lambda}\lambda, <^*)$, assuming that $\lambda^{<\lambda} = \lambda$.

Changing the bounding and dominating numbers

Strategy:

- Define a forcing to add a structure $D \in \mathcal{C}$ which dominates $\mathcal{C}^V$.
- Iterate the one-step forcing along a given poset Q with $b(Q) \geqslant \lambda^+$, adding a structure D_a for each $a \in Q$.
- Prove that $a \nless b$ implies that D_a does not embed into D_b.

Then $b(\mathcal{C}) = b(Q)$ and $d(\mathcal{C}) = d(Q)$ in the extension.

The Forcings

The conditions of the one-step forcing consist of

- an approximation of size $< \lambda$ to the dominating structure,
- partial embeddings of from $< \lambda$ structures into the approximation, and
- $< \lambda$ subsets of the approximation which cannot be extended.

The last condition ensures that the dominating structure is in the class.

Iteration

All our forcings are equivalent to forcings consisting of functions $p\colon \mathit{Ord} \to \lambda$ of size $< \lambda$ so that $p \leqslant q$ depends only on the common support and is absolute. These forcings are $< \lambda$-closed and λ^+-c.c.

The class of these forcings (and equivalent forcings) is closed under $< \lambda$-support (nonlinear) iteration.

Another way to see that the iterations are $< \lambda$-closed λ^+-c.c. is that they satisfy the properties for the iteration for Shelah's generalized Martin's axiom.

Covering properties

Let $b_{weak}(T_\lambda)$ and $d_{weak}(T_\lambda)$ denote the bounding and dominating numbers with respect to strict order-preserving maps (equal to $b(T_\lambda), d(T_\lambda)$ in our models).

Definition

Let cov_λ^+ (cov_λ^-) denote the least μ such that every (some) $\Pi_1^1 \backslash \Sigma_1^1$ subset of ${}^\lambda\lambda$ can be covered by μ many Σ_1^1 sets.

Then $b_{weak}(T_\lambda) \leqslant cov_\lambda^- \leqslant cov_\lambda^+ = d_{weak}(T_\lambda)$.

Question

Is always $b_{weak}(T_\lambda) = cov_\lambda^-$?

In the Cohen model $cov_\lambda^- = \lambda^+$ and $cov_\lambda^+ = 2^\lambda$.

Current work

Try to find a forcing for adding a dominating $c.c.c.$ poset of size λ.

Thank you for listening!