Point realizations of Boolean actions

Sławomir Solecki

University of Illinois at Urbana–Champaign

July 2012

Outline of Topics

1. Main problem
2. Some answers
3. Groups of isometries and unifying results
4. The borderline case: $C(M, \mathbb{T})$

Main problem

X a standard Borel space, for example, $X = [0, 1]$

X a standard Borel space, for example, $X = [0, 1]$

μ a Borel atomless probability measure on X

X a standard Borel space, for example, $X = [0, 1]$

μ a Borel atomless probability measure on X

$\mathrm{Aut}(\mu)$ = all measure preserving Boolean transformations of Borel/μ

Halmos–von Neumann:

given $f \in \mathrm{Aut}(\mu)$, $f \colon \mathrm{Borel}/\mu \to \mathrm{Borel}/\mu$,

Halmos–von Neumann:

given $f \in \mathrm{Aut}(\mu)$, $f\colon \mathrm{Borel}/\mu \to \mathrm{Borel}/\mu$,

there exists $F\colon X \to X$ a **Borel bijection**

Halmos–von Neumann:

given $f \in \mathrm{Aut}(\mu)$, $f\colon \mathrm{Borel}/\mu \to \mathrm{Borel}/\mu$,

there exists $F\colon X \to X$ a **Borel bijection** such that

$$f(A/\mu) = F[A]/\mu$$

for each Borel set $A \subseteq X$.

Halmos–von Neumann:

given $f \in \mathrm{Aut}(\mu)$, $f\colon \mathrm{Borel}/\mu \to \mathrm{Borel}/\mu$,

there exists $F\colon X \to X$ a **Borel bijection** such that

$$f(A/\mu) = F[A]/\mu$$

for each Borel set $A \subseteq X$.

f has a **point realization** F.

Topology on $\mathrm{Aut}(\mu)$ = the weakest topology making all the functions

$$\mathrm{Aut}(\mu) \ni f \to \mu(f(a) \triangle b) \in \mathbb{R}$$

continuous, where $a, b \in \mathrm{Borel}/\mu$.

Topology on $\mathrm{Aut}(\mu)$ = the weakest topology making all the functions

$$\mathrm{Aut}(\mu) \ni f \to \mu(f(a) \triangle b) \in \mathbb{R}$$

continuous, where $a, b \in \mathrm{Borel}/\mu$.

This is a **Polish group** (separable, completely metrizable) topology on $\mathrm{Aut}(\mu)$.

G a Polish group

G a Polish group

Consider a **continuous homomorphism**

$$\phi\colon G \to \mathrm{Aut}(\mu).$$

G a Polish group

Consider a **continuous homomorphism**

$$\phi\colon G \to \mathrm{Aut}(\mu).$$

Such a homomorphism ϕ can be viewed as an action

$$G \times (\mathrm{Borel}/\mu) \ni (g, a) \to \phi(g)(a) \in \mathrm{Borel}/\mu.$$

G a Polish group

Consider a **continuous homomorphism**

$$\phi\colon G \to \mathrm{Aut}(\mu).$$

Such a homomorphism ϕ can be viewed as an action

$$G \times (\mathrm{Borel}/\mu) \ni (g, a) \to \phi(g)(a) \in \mathrm{Borel}/\mu.$$

Actions of this sort are called **Boolean actions**.

G a Polish group

Consider a **continuous homomorphism**

$$\phi\colon G \to \mathrm{Aut}(\mu).$$

Such a homomorphism ϕ can be viewed as an action

$$G \times (\mathrm{Borel}/\mu) \ni (g, a) \to \phi(g)(a) \in \mathrm{Borel}/\mu.$$

Actions of this sort are called **Boolean actions**.

We will write $g(a)$ for $\phi(g)(a)$.

A **point realization** (or a **spatial model**) of a continuous homomorphism $G \to \mathrm{Aut}(\mu)$ (or a Boolean action) is a Borel action $G \times X \to X$

A **point realization** (or a **spatial model**) of a continuous homomorphism $G \to \mathrm{Aut}(\mu)$ (or a Boolean action) is a Borel action $G \times X \to X$ such that

$$g(A/\mu) = g[A]/\mu$$

for all $g \in G$ and $A \in \mathrm{Borel}$.

A **point realization** (or a **spatial model**) of a continuous homomorphism $G \to \mathrm{Aut}(\mu)$ (or a Boolean action) is a Borel action $G \times X \to X$ such that

$$g(A/\mu) = g[A]/\mu$$

for all $g \in G$ and $A \in \mathrm{Borel}$.

With notation $\phi\colon G \to \mathrm{Aut}(\mu)$ and $\alpha\colon G \times X \to X$,

A **point realization** (or a **spatial model**) of a continuous homomorphism $G \to \mathrm{Aut}(\mu)$ (or a Boolean action) is a Borel action $G \times X \to X$ such that

$$g(A/\mu) = g[A]/\mu$$

for all $g \in G$ and $A \in \mathrm{Borel}$.

With notation $\phi\colon G \to \mathrm{Aut}(\mu)$ and $\alpha\colon G \times X \to X$, the above equality says

$$\phi(g)(A/\mu) = \{\alpha(g,x)\colon x \in A\}/\mu.$$

Main question:

Main question:

For what Polish groups G the following holds

Main question:

For what Polish groups G the following holds
each continuous homomorphism $G \to \mathrm{Aut}(\mu)$ (Boolean action) has a point realization?

Some answers

The bad side

Some Polish groups have Boolean actions without point realizations.

Some Polish groups have Boolean actions without point realizations.

Vershik '87

Some Polish groups have Boolean actions without point realizations.

Vershik '87

Becker '02:
$G =$ measure classes of measurable subsets of $[0, 1]$ with symmetric difference as group operation

Some Polish groups have Boolean actions without point realizations.

Vershik '87

Becker '02:
$G =$ measure classes of measurable subsets of $[0, 1]$ with symmetric difference as group operation

Glasner–Tsirelson–Weiss '05:
$G =$ measure classes of measurable functions $[0, 1] \to \mathbb{T}$ with pointwise addition as group operation and with convergence in measure

The good side

Recall that S_∞ is the group of all permutations of $\mathbb{N}$ with composition and the topology of pointwise convergence.

Recall that S_∞ is the group of all permutations of $\mathbb{N}$ with composition and the topology of pointwise convergence.

Mackey '62:
If G is locally compact, then point realizations exist.

Recall that S_∞ is the group of all permutations of $\mathbb{N}$ with composition and the topology of pointwise convergence.

Mackey '62:
If G is locally compact, then point realizations exist.

Glasner–Weiss '05:
If G is a closed subgroup of S_∞, then point realizations exist.

Recall that S_∞ is the group of all permutations of $\mathbb{N}$ with composition and the topology of pointwise convergence.

Mackey '62:
If G is locally compact, then point realizations exist.

Glasner–Weiss '05:
If G is a closed subgroup of S_∞, then point realizations exist.

The proofs of these two results were very different.

Groups of isometries and unifying results

Groups of isometries

X a metric space

X a metric space
$\mathrm{Iso}(X)$ the group of all isometries of X with composition and pointwise convergence

X a metric space
$\mathrm{Iso}(X)$ the group of all isometries of X with composition and pointwise convergence

G a Polish group

X a metric space
$\mathrm{Iso}(X)$ the group of all isometries of X with composition and pointwise convergence

G a Polish group
G is a **Polish group of isometries of** X if G is a subgroup of $\mathrm{Iso}(X)$ as a topological group.

Groups of isometries of locally compact metric spaces were studied by **Gao–Kechris**.

Groups of isometries of locally compact metric spaces were studied by **Gao–Kechris**. Examples: locally compact groups, closed subgroups of S_∞, **closed subgroups of countable products of locally compact groups**

Groups of isometries of locally compact metric spaces were studied by **Gao–Kechris**. Examples: locally compact groups, closed subgroups of S_∞, **closed subgroups of countable products of locally compact groups**

Malicki–S.: locally compact groups = groups of isometries of proper metric spaces

The unifying result

Theorem (Kwiatkowska–S.)

Let G be a Polish group of isometries of a locally compact metric space. Then each continuous homomorphism $G \to \mathrm{Aut}(\mu)$ has a point realization.

Theorem (Kwiatkowska–S.)

Let G be a Polish group of isometries of a locally compact metric space. Then each continuous homomorphism $G \to \mathrm{Aut}(\mu)$ has a point realization.

The result unifies the theorems of Mackey and Glasner–Weiss.

Theorem (Kwiatkowska–S.)

Let G be a Polish group of isometries of a locally compact metric space. Then each continuous homomorphism $G \to \mathrm{Aut}(\mu)$ has a point realization.

The result unifies the theorems of Mackey and Glasner–Weiss.

New cases: closed subgroups of countable products of locally compact groups.

We need a new characterization of groups of isometries of locally compact metric spaces.

We need a new characterization of groups of isometries of locally compact metric spaces.

Recall that for H a subgroup of G

We need a new characterization of groups of isometries of locally compact metric spaces.

Recall that for H a subgroup of G

$$N(H) = \{g \in G \colon gHg^{-1} = H\}.$$

We need a new characterization of groups of isometries of locally compact metric spaces.

Recall that for H a subgroup of G

$$N(H) = \{g \in G \colon gHg^{-1} = H\}.$$

Theorem (Kwiatkowska–S.)

Let G be a Polish group. Then G is a group of isometries of a locally compact metric space if and only if

We need a new characterization of groups of isometries of locally compact metric spaces.

Recall that for H a subgroup of G

$$N(H) = \{g \in G \colon gHg^{-1} = H\}.$$

Theorem (Kwiatkowska–S.)

Let G be a Polish group. Then G is a group of isometries of a locally compact metric space if and only if for each $U \ni 1$ open there exists $H \subseteq U$ a closed subgroup of G such that

We need a new characterization of groups of isometries of locally compact metric spaces.

Recall that for H a subgroup of G

$$N(H) = \{g \in G \colon gHg^{-1} = H\}.$$

Theorem (Kwiatkowska–S.)

Let G be a Polish group. Then G is a group of isometries of a locally compact metric space if and only if for each $U \ni 1$ open there exists $H \subseteq U$ a closed subgroup of G such that

G/H is a locally compact space and

We need a new characterization of groups of isometries of locally compact metric spaces.

Recall that for H a subgroup of G

$$N(H) = \{g \in G \colon gHg^{-1} = H\}.$$

Theorem (Kwiatkowska–S.)

Let G be a Polish group. Then G is a group of isometries of a locally compact metric space if and only if for each $U \ni 1$ open there exists $H \subseteq U$ a closed subgroup of G such that

G/H is a locally compact space and

$N(H)$ is open.

About the proof of the second theorem

Gao–Kechris:

G a Polish group.

Gao–Kechris:

G a Polish group.

G is an isometry group of a locally compact metric space if and only if

Gao–Kechris:

G a Polish group.

G is an isometry group of a locally compact metric space if and only if G is a **closed subgroup** of a **countable product** of groups of the form

Gao–Kechris:

G a Polish group.

G is an isometry group of a locally compact metric space if and only if G is a **closed subgroup** of a **countable product** of groups of the form

$$S_\infty \ltimes H^{\mathbb{N}},$$

where H is locally compact and S_∞ acts by homomorphisms on $H^{\mathbb{N}}$ by permuting coordinates.

Lemma

The condition from the theorem is preserved under taking closed subgroups.

Lemma

The condition from the theorem is preserved under taking closed subgroups.

Proof uses

Lemma

The condition from the theorem is preserved under taking closed subgroups.

Proof uses Yamabe's theorem connecting locally compact groups with Lie groups (Hilbert's 5-th problem)

Lemma

The condition from the theorem is preserved under taking closed subgroups.

Proof uses Yamabe's theorem connecting locally compact groups with Lie groups (Hilbert's 5-th problem) and well behaved dimension on Lie groups.

The borderline case: $C(M, \mathbb{T})$

Groups of continuous functions and the theorem

Let M be a compact metric space.

Let M be a compact metric space.
Let $C(M, \mathbb{T})$ be the group of all continuous functions from M to $\mathbb{T}$ with pointwise multiplication and with the uniform convergence topology.

Let M be a compact metric space.
Let $C(M, \mathbb{T})$ be the group of all continuous functions from M to $\mathbb{T}$ with pointwise multiplication and with the uniform convergence topology.

$C([0,1], \mathbb{T})$ lies exactly between
$\{f : [0,1] \to \mathbb{T}$ measurable$\}$, which has non-point realizable Boolean actions

Let M be a compact metric space.
Let $C(M, \mathbb{T})$ be the group of all continuous functions from M to $\mathbb{T}$ with pointwise multiplication and with the uniform convergence topology.

$C([0,1], \mathbb{T})$ lies exactly between
$\{f : [0,1] \to \mathbb{T}$ measurable$\}$, which has non-point realizable Boolean actions
and
groups with the property from the second theorem, whose Boolean actions have point realizations.

Theorem (Moore–S.)

Let M be a compact uncountable metric space. The group $C(M, \mathbb{T})$ has a Boolean action that does not have a point realization.

An outline of proof in the case $M = 2^{\mathbb{N}}$

Identify $\mathbb{C}$ with $\mathbb{R}^2$.

Identify $\mathbb{C}$ with $\mathbb{R}^2$.

Let γ be the standard Gaussian measure on $\mathbb{C}$ with density

$$\frac{1}{2\pi} e^{-\frac{1}{2}(x_0^2+x_1^2)}$$

Note

Note γ is **preserved** under rotations of $\mathbb{C}$ by elements of $\mathbb{T}$

Note γ is **preserved** under rotations of $\mathbb{C}$ by elements of $\mathbb{T}$,

$$\pi\colon \mathbb{C} \times \mathbb{C} \ni (z_1, z_2) \to \frac{z_1 + z_2}{\sqrt{2}} \in \mathbb{C}$$

is **measure preserving** if $\mathbb{C} \times \mathbb{C}$ is taken with $\gamma \times \gamma$ and $\mathbb{C}$ with γ, Note γ is **preserved** under rotations of $\mathbb{C}$ by elements of $\mathbb{T}$,

$$\pi \colon \mathbb{C} \times \mathbb{C} \ni (z_1, z_2) \to \frac{z_1 + z_2}{\sqrt{2}} \in \mathbb{C}$$

is **measure preserving** if $\mathbb{C} \times \mathbb{C}$ is taken with $\gamma \times \gamma$ and $\mathbb{C}$ with γ, and

$$\iota \colon \mathbb{T} \ni z \to (z, z) \in \mathbb{T} \times \mathbb{T}$$

is a **continuous embedding**.

$$\begin{array}{c}(\mathbb{C}, \gamma)\\ \uparrow\\ \mathbb{T}\end{array}$$

$$\begin{array}{ccc} (\mathbb{C}, \gamma) & \xleftarrow{\ \pi\ } & (\mathbb{C}^2, \gamma^2) \\ \uparrow & & \uparrow \\ \mathbb{T} & \xrightarrow{\ \iota\ } & \mathbb{T}^2 \end{array}$$

$$
\begin{array}{ccccc}
(\mathbb{C}, \gamma) & \xleftarrow{\ \pi\ } & (\mathbb{C}^2, \gamma^2) & \xleftarrow{\ \pi^2\ } & (\mathbb{C}^4, \gamma^4) \\
\uparrow & & \uparrow & & \uparrow \\
\mathbb{T} & \xrightarrow{\ \iota\ } & \mathbb{T}^2 & \xrightarrow{\ \iota^2\ } & \mathbb{T}^4
\end{array}
$$

$$
\begin{array}{ccccccc}
(\mathbb{C}, \gamma) & \xleftarrow{\pi} & (\mathbb{C}^2, \gamma^2) & \xleftarrow{\pi^2} & (\mathbb{C}^4, \gamma^4) & \xleftarrow{\pi^3} & (\mathbb{C}^8, \gamma^8) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\mathbb{T} & \xrightarrow{\iota} & \mathbb{T}^2 & \xrightarrow{\iota^2} & \mathbb{T}^4 & \xrightarrow{\iota^3} & \mathbb{T}^8
\end{array}
$$

$$
\begin{array}{ccccc}
(\mathbb{C}, \gamma) & \xleftarrow{\ \pi\ } & (\mathbb{C}^2, \gamma^2) & \xleftarrow{\ \pi^2\ } & \cdots \varprojlim (\mathbb{C}^{2^n}, \gamma^{2^n}) \\
\uparrow & & \uparrow & & \uparrow \\
\mathbb{T} & \xrightarrow{\ \iota\ } & \mathbb{T}^2 & \xrightarrow{\ \iota^2\ } & \cdots \varinjlim \mathbb{T}^{2^n}
\end{array}
$$

$$\begin{array}{ccccccc}
(\mathbb{C},\gamma) & \xleftarrow{\pi} & (\mathbb{C}^2,\gamma^2) & \xleftarrow{\pi^2} & \cdots \varprojlim(\mathbb{C}^{2^n},\gamma^{2^n}) & =\!= & (\mathbb{C}^\infty,\gamma^\infty)\\
\uparrow & & \uparrow & & \uparrow & & \\
\mathbb{T} & \xrightarrow{\iota} & \mathbb{T}^2 & \xrightarrow{\iota^2} & \cdots \varinjlim \mathbb{T}^{2^n} & \xrightarrow{\subseteq} & C(2^{\mathbb{N}},\mathbb{T})
\end{array}$$

$$
\begin{array}{ccccccc}
(\mathbb{C}, \gamma) & \xleftarrow{\;\pi\;} & (\mathbb{C}^2, \gamma^2) & \xleftarrow{\;\pi^2\;} & \cdots \varprojlim (\mathbb{C}^{2^n}, \gamma^{2^n}) & = & (\mathbb{C}^\infty, \gamma^\infty) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\mathbb{T} & \xrightarrow{\;\iota\;} & \mathbb{T}^2 & \xrightarrow{\;\iota^2\;} & \cdots \varinjlim \mathbb{T}^{2^n} & \xrightarrow{\;\subseteq\;} & C(2^{\mathbb{N}}, \mathbb{T})
\end{array}
$$

$$\begin{array}{ccccccc}
(\mathbb{C}, \gamma) & \xleftarrow{\ \pi\ } & (\mathbb{C}^2, \gamma^2) & \xleftarrow{\ \pi^2\ } & \cdots \varprojlim (\mathbb{C}^{2^n}, \gamma^{2^n}) & = & (\mathbb{C}^\infty, \gamma^\infty) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\mathbb{T} & \xrightarrow{\ \iota\ } & \mathbb{T}^2 & \xrightarrow{\ \iota^2\ } & \cdots \varinjlim \mathbb{T}^{2^n} & \xrightarrow{\ \subseteq\ } & C(2^{\mathbb{N}}, \mathbb{T})
\end{array}$$

We get a Boolean action of $C(2^{\mathbb{N}}, \mathbb{T})$ on the probability measure space $(\mathbb{C}^\infty, \gamma^\infty)$.

The proof of the following result is important for the proof of non-point realizability:

The proof of the following result is important for the proof of non-point realizability:

if $a \in \mathbb{R}$ and $B \subseteq \mathbb{R}^{\mathbb{N}}$ is a Borel set of positive $\gamma^{\mathbb{N}}$-measure, then

$$\gamma^{\mathbb{N}}(\sqrt{1+a^2}B + ay) > 0, \quad \text{for } \gamma^{\mathbb{N}}\text{-a.e. } y \in \mathbb{R}^{\mathbb{N}}.$$