Constructing Lindelöf, non D-spaces

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- compact and Lindelöf spaces

Definition

*An **open neighborhood assignment** (ONA, in short) on a space (X, τ) is a map $U : X \to \tau$ such that $x \in U(x)$ for every $x \in X$.*

X is **compact** $\Leftrightarrow$ for every ONA U on X there is a **finite** $D \subseteq X$ such that $X = \bigcup U[D]$

- generalization: **finite** $D \longrightarrow$ **locally finite** D.

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D-spaces

Definition (E. van Douwen)

*X is a **D-space** iff for every neighborhood assignment U, there is a **closed and discrete** $D \subseteq X$ (i.e. locally finite) such that $X = \bigcup U[D]$.*

- every σ-compact or metric space is a D-space
- ω_1 **is not a D-space** (every closed discrete set is finite, however non compact)

Main Problem (E. van Douwen, 1975)

*Is it true, that **every Lindelöf space is a D-space**?*

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Recent progress

Which covering properties imply D?

Definition

X is Menger iff for every sequence of open covers $\{\mathcal{U}_n : n \in \omega\}$ there are $\mathcal{V}_n \in [\mathcal{U}_n]^{<\omega}$ such that $\bigcup\{\cup\mathcal{V}_n : n \in \omega\} = X$.

Theorem (L. Aurichi)

Every Menger space is a D-space.

Corollary

Every Lindelöf space of size less than $\mathfrak{d}$ is a D-space.

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Theorem (D. Repovŝ, L. Zdomskyy)

*Consistently, all **(sub)paracompact spaces of size** $\leq \omega_1$ are D-spaces.*

Definition

X *is **productively Lindelöf** iff* $X \times Y$ *is Lindelöf for every Lindelöf space* Y.

Theorem (F. Tall)

*Under CH , **regular, productively Lindelöf spaces** are D-spaces.*

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Main results

Consistent counterexamples

Theorem (S.-Szeptycki)

Under $\diamondsuit$, there is a ***Hausdorff, hereditarily Lindelöf non D-space*** *of size ω_1.*

Remark: it is consistent that $\clubsuit$ holds, 2^ω is arbitrarily large and every T_1 Lindelöf space of size less than 2^ω is a D-space.

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About the construction

of a Hasudorff, hereditarily Lindelöf non D-space

The underlying set for our space (X, τ) is ω_1 and we **inductively build**

- $\mathcal{U} = \{U_\gamma : \gamma < \omega_1\}$ subsets of ω_1 such that $\gamma \in U_\gamma$,
- a map φ from ω_1 to a second countable Hausdorff space (Y, ρ).

τ is generated by $\mathcal{U} \cup \varphi^{-1}[\rho]$ as a subbase. We want:

- $\gamma \to U_\gamma$ to show that X is not a D-space,
- the proper inductive hypothesises to guarantee that X is hereditarily Lindelöf.

Note: X will be Hausdorff as we "refine" by ρ.

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About the construction

Building $\gamma \to U_\gamma$

List $[\omega_1]^{\leq\omega}$ as $\{C_\alpha\}_{\alpha<\omega_1}$. In the αth step of the induction, we will

- **construct the initial segment $U_\gamma \cap (\alpha+1)$** for $\gamma \leq \alpha$,
- **decide if $\alpha \in U_\gamma$ or $\alpha \notin U_\gamma$** for $\gamma \leq \alpha$,
- if C_α is closed discrete (where?) then $\alpha \notin U_\gamma$ for $\gamma \in C_\alpha$.

Use the $\diamondsuit$-sequence to

- **guess countable segments of open covers from the final topology**,
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a seemingly **CRAZY TECHNICAL** inductive hypothesises.

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Further properties

and corresponding questions

Theorem

The above constructed space X has the following further properties:

- *X is **dually second countable**,*
- ***every finite power of X is Lindelöf**.*

Theorem (S.-Szeptycki)

*Under $\diamondsuit$, there is a Hausdorff, hereditarily Lindelöf **non D-space which is the union of two D-spaces**.*

- Are there regular/0-dimensional examples? In ZFC?

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Various open problems

á la Gary Gruenhage, A survey of D-spaces, Contemporary Math. 533(2011),13–28.

The following are still unknown:

- Let S be the **Sorgenfrey line. Is S^ω a D-space**?
- Is every butterfly space over a separable metric space a D-space?
- **Is every paracompact, monotonically normal space a D-space**?
- Suppoes that X is regular and X^ω is hereditarily Lindelöf. Is X a D-space?
- Is every hereditarily Lindelöf/MN/σ-discrete space dually discrete?
- Suppose that **$A \times B$ is Lindelöf for all $A \subseteq X$ and $B \subseteq Y$. Is $X \times Y$ hereditarily Lindelöf?** If $X = Y$?

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- **Is every paracompact, monotonically normal space a D-space**?
- Suppoes that X is regular and X^ω is hereditarily Lindelöf. Is X a D-space?
- Is every hereditarily Lindelöf/MN/σ-discrete space dually discrete?
- Suppose that $A \times B$ **is Lindelöf for all $A \subseteq X$ and $B \subseteq Y$. Is $X \times Y$ hereditarily Lindelöf?** If $X = Y$?

Various open problems

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The End

Thank you for your attention!