On a Topological Choice Principle by Murray Bell

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Statement of Bell's choice principle and open problem

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- **Both questions of the problem are still unresolved**.

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  - On the other hand, BPI does not imply AC in ZF (J. D. Halpern and A. Lévy, 1967, [2]).

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- One might think of extending (using some weak form of AC) a definable compact T_1 topology on A_i to a compact Hausdorff topology. But, even in ZFC, this may not be feasible (e.g., the one-point compactification of $\mathbb{Q}$ with its standard topology).

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- One might think of extending (using some weak form of AC) a definable compact T_1 topology on A_i to a compact Hausdorff topology. But, even in ZFC, this may not be feasible (e.g., the one-point compactification of $\mathbb{Q}$ with its standard topology).
  - Close to this, Herrlich and Keremedis, 2011, [3], showed that if *for every set X, every compact R_1 topology on X (i.e., its T_0-identification is Hausdorff) can be extended to a compact Hausdorff topology*, then (C) holds.

- Some form of choice could be derived from (C), if we could decide whether some points in A_i (with an assigned, by (C), compact Hausdorff topology T_i) satisfy a certain (topological) property P_i, while others don't satisfy P_i. For example,

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      However, in ZF, a countable compact Hausdorff space may fail to be metrizable (K. Keremedis, E. Tachtsis, 2007, [8]).

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- Due to the non-constructive character of (C) and due to the fact that we may know nothing on the nature of the sets in an infinite family, upon which (C) is applied, it seems reasonable to think that further suitable assumptions must be added to (C) in order to derive certain choice forms.

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- The answer is an emphatic NO!

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- $\mathrm{PAC}_n^{\aleph_0}$ (where $n \in \mathbb{N}$): *For every denumerable family* $\mathcal{A}$ *of non-empty sets each with at most* n *elements, there is an infinite subfamily of* $\mathcal{A}$ *with a choice function.*

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Theorem

The following implications hold in ZF:

1. *(C) + UF(ω) implies $\mathrm{PKW}^{\aleph_0}$.*
2. *(C) + UF(ω) implies "For every integer $n \geq 2$, $\mathrm{PAC}_n^{\aleph_0}$".*

Proof.

(1) By way of contradiction, assume the existence of a disjoint family $\mathcal{A} = \{A_i : i \in \omega\}$, where $\forall i \in \omega$, $|A_i| \geq 2$, without a partial Kinna-Wagner (pKW) function. For each $X \subseteq \bigcup \mathcal{A}$, let T_X be a compact Hausdorff topology on X. By induction we define a partial choice function for $\mathcal{A}$.

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- Assume that we have chosen integers $n_0 < n_1 < \ldots < n_k$ and elements $y_{n_i} \in A_{n_i}$ for $i = 0, 1, \ldots, k$.

- Consider the compact Hausdorff space $(X_{k+1}, T_{X_{k+1}})$, where $X_{k+1} = (\bigcup \mathcal{A})\backslash(\bigcup_{i \leq n_k} A_i)$. The set

$$\mathcal{H}_{k+1} = \{\bigcup\{A_n : n \in F\backslash(n_k + 1)\} : F \in \mathcal{F}_0\}$$

is a base for some free ultrafilter $\mathcal{G}_{k+1}$ on X_{k+1}. Hence, there is a unique element $y_{n_{k+1}} \in A_{n_{k+1}}$, where n_{k+1} is an integer greater than n_k, such that $\mathcal{G}_{k+1} \to y_{n_{k+1}}$.

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(2) Use part 1 and mathematical induction. □

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Let $\mathcal{N}$ be the second Fraenkel permutation model: The set of atoms $A = \bigcup\{A_n : n \in \omega\}$, where $\forall n \in \omega$, $|A_n| = 2$. The group G of permutations of A consists of all π such that $\forall n \in \omega$, $\pi(A_n) = A_n$. The normal ideal of supports is $[A]^{<\omega}$. $\mathcal{N}$ is the FM model determined by G and $[A]^{<\omega}$. The following facts about $\mathcal{N}$ are well-known (Howard-Rubin [6], Jech [7]):

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Therefore, $\mathcal{N} \models \neg$(C). □

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4. *(Tachtsis, 2010, [11])* $\forall X$, *the Cantor cube* 2^X *is sequentially accumulation point compact. In particular,* $2^{\mathbb{R}}$ *is s.a.p.c.*

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It's fairly easy to see that:

- In ZF, CBPI implies UF(ω).
- In ZF, (CBPI restricted to countable sets) iff UF(ω).

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(C) + "For a product of countably many compact Hausdorff spaces canonical projections are closed" implies $AC^{\aleph_0}$ (AC restricted to countable families of non-empty sets).

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- Note that item 4 of the previous theorem is in **striking** contrast with the corresponding ZF-equivalence "AC iff (C) + BPI".

- A Hausdorff space (X, T) is called *effectively normal* if there is a function F such that for every pair (A, B) of disjoint closed sets in X, $F(A, B) = (C, D)$ where C and D are disjoint open sets such that $A \subseteq C$ and $B \subseteq D$. F is called a *normality operator*.

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- MC iff every normal space is effectively normal. Hence, MC implies every compact Hausdorff space is effectively normal.
- "Every compact Hausdorff space is effectively normal" is not a theorem of ZF. In particular, it implies E. van Douwen's choice principle.
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Proof. (1) It suffices to show that every compact Hausdorff space (X, T), where X is a countable union of finite sets, has at least one isolated point. Fix such a space (X, T), where $X = \bigcup_{n \in \omega} X_n$, $|X_n| < \aleph_0$, and let F be a normality operator on X. By way of contradiction assume that X is dense-in-itself.

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  - Keypoint for the above construction: Using F, **we can effectively determine**, for every pair (A, B) of disjoint finite subsets of X, two open sets U and V such that $A \subseteq U$, $B \subseteq V$ and $\mathrm{cl}_X(U) \cap \mathrm{cl}_X(V) = \emptyset$.

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- For each $f \in {}^{\omega}2$, let $G_f = \bigcap_{n\in\omega} \mathrm{cl}_X(B_{f\restriction n})$. By compactness of X, $G_f \neq \emptyset$. Let also, for $f \in {}^{\omega}2$, $n_f = \min\{n \in \omega : G_f \cap X_n \neq \emptyset\}$.

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- Define the function $H : {}^{\omega}2 \to \bigcup_{n\in\omega} \mathcal{P}(X_n)$, by letting $H(f) = G_f \cap X_{n_f}$. Then H is 1-1, hence ${}^{\omega}2$ is countable, being a countable union of finite wosets. A contradiction.

Therefore, X is scattered as required.

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  - For distinct $x, y \in X$, let $F(\{x\},\{y\}) = (U_x^y, V_y^x)$. Then $\mathcal{C} = \{U_x^y : x, y \in X, x \neq y\} \cup \{V_y^x : x, y \in X, x \neq y\}$ is countable, hence $\mathcal{B} = \{\bigcap \mathcal{D} : D \in [\mathcal{C}]^{<\omega}\}$ is also countable. Furthermore, $\mathcal{B}$ is a base for the topology T on X.

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- By Urysohn's Metrization Theorem (which is provable in ZF, C. Good and I. Tree, 1995, [1]), X is metrizable.
- Since, in ZF, every compact metrizable space with a well-ordered dense subset is a *Baire space* (the intersection of each countable family of dense open sets is dense), X is scattered. □

Theorem

In ZF, (C) + "Every compact Hausdorff space is effectively normal" implies:

- *Every family* $\mathcal{A} = \{A_i : i \in I\}$, *where for each* $i \in I$, A_i *can be written as a countable union of non-empty finite sets, has a multiple choice function.*

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Proof. For each $i \in I$, let T_i be a compact Hausdorff topology on A_i. By the Lemma, each A_i is scattered. Let $\beta_i = \alpha_i + 1$ be the height of A_i. Then for each $i \in I$, the Cantor-Bendixson derivative $(A_i)_{\alpha_i}$ is a non-empty finite subset of A_i. Hence, $f = \{(i, (A_i)_{\alpha_i}) : i \in I\}$ is a MC function for $\mathcal{A}$. □

Theorem

For a countable compact Hausdorff space (X, T), the following are equivalent:

- *X is metrizable,*
- *X is second countable,*
- *X (topologically) embeds as a closed subspace of $[0,1]^\omega$,*
- *X is effectively normal.*

- Since "Every countable compact Hausdorff space is metrizable" is not a theorem of ZF (Keremedis and Tachtsis, 2007, [8]), it follows that neither "Every countable compact Hausdorff space is effectively normal" is provable from the ZF axioms alone.

Corollary

In ZF, (C) + "Every countable compact Hausdorff space is effectively normal" implies each one of the following statements:

- $\mathbb{R}$ *cannot be written as a countable union of countable sets.*
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- The assumption of (C), in the previous theorem, cannot be dropped; In the second Fraenkel model $\mathcal{N}$, every compact Hausdorff space is effectively normal (since $\mathcal{N} \models$ MC), whereas there is a countable family of pairs in $\mathcal{N}$ without a partial choice function.

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- AC($\aleph_0, \mathbb{R}$) implies that every countable compact Hausdorff space is metrizable, hence scattered (Keremedis and Tachtsis, 2007, [8]). Thus, we may define a multiple choice function for $\mathcal{V} = \{[x] : x \in \mathbb{R}\}$, hence a choice function f for $\mathcal{V}$, since $\forall x \in \mathbb{R}$, $[x] \subseteq \mathbb{R}$.

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- AC($\aleph_0, \mathbb{R}$) implies that the Lebesgue measure is σ-additive, hence following the well-known proof of the existence of a non-measurable set of reals, one verifies that $E = \{f([x]) : x \in \mathbb{R}\}$ is non-measurable. □

Corollary

(C) fails in the following ZF-models:

- *Solovay's model ($\mathcal{M}5(\aleph)$ in Howard-Rubin [6]).*
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- In $\mathcal{M}5(\aleph)$, $\mathrm{AC}(\aleph_0, \mathbb{R})$ holds but every set of reals is Lebesgue measurable. Hence, (C) fails in $\mathcal{M}5(\aleph)$.

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  - AC for well orderable families of non-empty sets, hence $\mathrm{AC}(\aleph_0, \mathbb{R})$, holds in $\mathcal{M}2$.
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  If (C) were true in $\mathcal{M}2$, then using ideas from the proof of the previous Theorem we would obtain that the family $\mathcal{B} = \{[A] : A \subseteq \omega\}$ admits a choice set, and since $\mathcal{P}(\omega)$ is linearly orderable, a choice set for $\mathcal{A}$ would exist in $\mathcal{M}2$, which is impossible. Hence, (C) cannot hold in Feferman's model.

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