A Descriptive View of Unitary Group Representations

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Finite Dimensional Representations

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Two representations $\varphi : G \to GL_n(\mathbb{C})$ *and* $\psi : G \to GL_m(\mathbb{C})$ *are equivalent if* $n = m$ *and there exists* $A \in GL_n(\mathbb{C})$ *such that*

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Definition

The representation $\varphi : G \to GL_n(\mathbb{C})$ is irreducible if there are no nontrivial proper G-invariant subspaces $0 < W < \mathbb{C}^n$.

Finite Dimensional Representations

Theorem

If G is a finite group, then:

(i) *G has finitely many irreducible representations.*

(ii) *Every representation of G is uniquely expressible as a direct sum of irreducible representations.*

Definition

If $\varphi : G \to GL_n(\mathbb{C})$ *and* $\psi : G \to GL_m(\mathbb{C})$ *are representations, then the direct sum* $(\varphi \oplus \psi) : G \to GL_{n+m}(\mathbb{C})$ *is defined by*

$$g \mapsto \begin{pmatrix} \varphi(g) & \mathbf{0} \\ \mathbf{0} & \psi(g) \end{pmatrix}$$

Finite Dimensional Representations

Definition

- *If G is a finite group, then a* *unitary representation* *of G is a homomorphism $\varphi : G \to U_n(\mathbb{C})$ for some $n \geq 1$.*
- *Here the* *unitary group* *$U_n(\mathbb{C})$ is the subgroup of $GL_n(\mathbb{C})$ which preserves the inner product*

$$\langle u, v \rangle = u_1 \bar{v}_1 + \cdots + u_n \bar{v}_n.$$

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Theorem

If G is a finite group, then:

(i) *Every representation of G is equivalent to a unitary representation.*

(ii) *The unitary representations $\varphi, \psi : G \to U_n(\mathbb{C})$ are equivalent iff there exists $A \in U_n(\mathbb{C})$ such that*

$$\psi(g) = A\,\varphi(g)\,A^{-1} \qquad \text{for all } g \in G.$$

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Example

Consider the Hilbert space

$$\ell^2(G) = \{ \, (a_g) \in \mathbb{C}^G \mid \sum |a_g|^2 < \infty \, \}.$$

Then we can define a unitary representation $\varphi : G \to U(\, \ell^2(G)\,)$ by

$$(\, a_x \,) \overset{\varphi(g)}{\mapsto} (\, a_{g^{-1}x} \,).$$

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Definition

Two representations $\varphi : G \to U(\mathcal{H})$ *and* $\psi : G \to U(\mathcal{H})$ *are* *unitarily equivalent* *if there exists* $A \in U(\mathcal{H})$ *such that*

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- *Can we classify the* ***irreducible*** *unitary representations of G up to unitary equivalence?*

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Problem

- *Can we classify the* *irreducible* *unitary representations of* G *up to unitary equivalence?*
- *Can we classify* *arbitrary* *unitary representations of* G *via "suitable decompositions" into irreducible representations?*

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- The irreducible unitary representations of $\mathbb{Z}$ are

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- The multiplicity-free unitary representations of $\mathbb{Z}$ can be parameterized by the Borel probability measures μ on $\mathbb{T}$ so that the following are equivalent:
  - (i) the representations φ_μ, φ_ν are unitarily equivalent;
  - (ii) the measures μ, ν have the same null sets.

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- The set $\mathrm{Rep}_n(G) \subseteq U(\mathcal{H})^G$ of unitary representations is a closed subspace and hence $\mathrm{Rep}_n(G)$ is a Polish space.

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- Then $U(\mathcal{H})$ is a Polish group and hence $U(\mathcal{H})^G$ with the product topology is a Polish space.
- The set $\mathrm{Rep}_n(G) \subseteq U(\mathcal{H})^G$ of unitary representations is a closed subspace and hence $\mathrm{Rep}_n(G)$ is a Polish space.
- The set $\mathrm{Irr}_n(G)$ of irreducible representations is a G_δ subset of $\mathrm{Rep}_n(G)$ and hence $\mathrm{Irr}_n(G)$ is also a Polish space.

Borel equivalence relations

Definition

An equivalence relation E on a Polish space X is **Borel** *if E is a Borel subset of $X \times X$.*

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Theorem (Mackey)

The unitary equivalence relation $\approx_G$ on $\mathrm{Irr}_n(G)$ is an F_σ equivalence relation.

Theorem (Hjorth-Törnquist)

The unitary equivalence relation $\approx_G^+$ on $\mathrm{Rep}_n(G)$ is an $F_{\sigma\delta}$ equivalence relation.

Smooth vs Nonsmooth

Definition (Mackey)

The Borel equivalence relation E on the Polish space X is smooth if there exists a Borel map $\varphi : X \to \mathbb{C}$ such that

$$x \, E \, y \iff \varphi(x) = \varphi(y).$$

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Corollary

If G is a countable group, then unitary equivalence for finite dimensional irreducible unitary representations of G is smooth.

The Glimm-Thoma Theorem

Theorem (Glimm-Thoma)

If G is a countable group, then the following are equivalent:

(i) *G is not abelian-by-finite.*

(ii) *G has an infinite dimensional irreducible representation.*

(iii) *The unitary equivalence relation $\approx_G$ on the space $\mathrm{Irr}_\infty(G)$ of infinite dimensional irreducible unitary representations of G is not smooth.*

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Question

Does this mean that we should abandon all hope of finding a "satisfactory classification" for the irreducible unitary representations of the non-(abelian-by-finite) groups?

Borel reductions

Definition

Let E, F be Borel equivalence relations on the Polish spaces X, Y.

- $E \leq_B F$ *if there exists a Borel map* $\varphi : X \to Y$ *such that*

$$x \, E \, y \iff \varphi(x) \, F \, \varphi(y).$$

  In this case, f is called a **Borel reduction** *from E to F.*
- $E \sim_B F$ *if both* $E \leq_B F$ *and* $F \leq_B E$.
- $E <_B F$ *if both* $E \leq_B F$ *and* $E \nsim_B F$.

The Glimm-Effros Dichotomy

Theorem (Harrington-Kechris-Louveau)

If E is a Borel equivalence relation on the Polish space X, then exactly one of the following holds:

(i) *E is smooth; or*

(ii) $E_0 \leq_B E$.

Definition

E_0 is the Borel equivalence relation on $2^{\mathbb{N}}$ defined by:

$$x \, E_0 \, y \quad \Longleftrightarrow \quad x_n = y_n \quad \text{for all but finitely many } n.$$

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Example

Baer's classification of the rank 1 torsion-free abelian groups is essentially a Borel reduction to E_0.

When it's bad, it's worse ...

Theorem (Hjorth 1997)

If the countable group G is not abelian-by-finite , then there exists a $U(\mathcal{H})$-invariant Borel subset $X \subseteq \mathrm{Irr}_\infty(G)$ such that the unitary equivalence relation $\approx_G \restriction X$ is turbulent.

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Remark

This is a much more serious obstruction to the existence of a "satisfactory classification" of the irreducible unitary representations of G.

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The Main Question (Thomas)

Do there exist countable groups G, H such that

(i) G, H are not abelian-by-finite; and

(ii) $\approx_G$, $\approx_H$ are not Borel bireducible?

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The Main Conjecture (Thomas)

If G is a nonabelian free group and H is a "suitably chosen" amenable group, then $\approx_H <_B \approx_G$.

Nonabelian free groups

Notation

$\mathbb{F}_n$ denotes the free group on n generators for $n \in \mathbb{N}^+ \cup \{\infty\}$.

Observation

If G is any countable group, then $\approx_G$ is Borel reducible to $\approx_{\mathbb{F}_\infty}$.

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Proof.

If $\theta : \mathbb{F}_\infty \to G$ is a surjective homomorphism, then the induced map

$$\mathrm{Irr}_\infty(G) \to \mathrm{Irr}_\infty(\mathbb{F}_\infty)$$
$$\varphi \mapsto \varphi \circ \theta$$

is a Borel reduction from $\approx_G$ to $\approx_{\mathbb{F}_\infty}$. □

Nonabelian free groups

Theorem

$\approx_{\mathbb{F}_\infty}$ *is Borel reducible to* $\approx_{\mathbb{F}_2}$.

Nonabelian free groups

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Sketch Proof (Warning: do not attempt to understand!)

If $f : \mathbb{N} \to \mathbb{N}$ be a suitably fast growing function, then we can induce representations from

$$\mathbb{F}_\infty = \langle\, a^{f(n)}\, b\, a^{-f(n)} \mid n \in \mathbb{N} \,\rangle \leqslant N = \langle\, a^m\, b\, a^{-m} \mid m \in \mathbb{Z} \,\rangle$$

to the free group $\mathbb{F}_2 = \langle\, a, b \,\rangle$. □

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to the free group $\mathbb{F}_2 = \langle\, a, b \,\rangle$. □

Question

- *Does $H \leqslant G$ imply that $\approx_H$ is Borel reducible to $\approx_G$?*
- *In particular, is $\approx_{\mathbb{F}_2}$ Borel reducible to $\approx_{SL(3,\mathbb{Z})}$?*

A suitably chosen amenable group?

Definition

A countable group G is amenable if there exists a left-invariant finitely additive probability measure $\mu : \mathcal{P}(G) \to [0, 1]$.

Some Suitably Chosen Amenable Groups?

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Some Suitably Chosen Amenable Groups?

- *The direct sum* $\bigoplus_{n \in \mathbb{N}} \mathrm{Sym}(3)$ *of countably many copies of* $\mathrm{Sym}(3)$.
- *A countably infinite extra-special p-group P; i.e. $P' = Z(P)$ is cyclic of order p and $P/Z(P)$ is elementary abelian p-group.*

Perhaps not quite as expected ...

- The following result is an immediate consequence of the work of Glimm (1961) and Elliot (1977).

Theorem

Suppose that the countable group G is not abelian-by-finite. If H is any countable locally finite group, then $\approx_H$ is Borel reducible to $\approx_G$.

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- The following result is an immediate consequence of the work of Glimm (1961) and Elliot (1977).

Theorem

Suppose that the countable group G is not abelian-by-finite. If H is any countable locally finite group, then $\approx_H$ is Borel reducible to $\approx_G$.

Corollary

If G, H are countable locally finite groups, neither of which is abelian-by-finite, then $\approx_G$ and $\approx_H$ are Borel bireducible.

The reduced C^*-algebra

Definition

If G is a countably infinite group, then the left regular representation

$$\lambda : G \to U(\,\ell^2(G)\,)$$

extends to an injective $*$-homomorphism of the group algebra

$$\lambda : \mathbb{C}[\,G\,] \to \mathcal{L}(\,\ell^2(G)\,).$$

The reduced C^*-algebra $C^*_\lambda(\,G\,)$ is the completion of $\mathbb{C}[\,G\,]$ with respect to the norm $||x||_r = ||\lambda(x)||_{\mathcal{L}(\,\ell^2(G)\,)}$.

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Remark

If G is amenable, then there is a canonical correspondence between the irreducible representations of G and $C^*_\lambda(\,G\,)$.

Approximately finite dimensional C^*-algebras

Definition

A C^*-algebra A is approximately finite dimensional if $A = \overline{\bigcup_{n\in\mathbb{N}} A_n}$ is the closure of an increasing chain of finite dimensional sub-C^*-algebras A_n.

Example

If $G = \bigcup_{n\in\mathbb{N}} G_n$ is a locally finite group, then $C^*_\lambda(G) = \overline{\bigcup_{n\in\mathbb{N}} \mathbb{C}[G_n]}$ is approximately finite dimensional.

Elliot's Theorem

- Extending Glimm's Theorem, Elliot proved:

Theorem (Elliot 1977)

If $\mathcal{A}$ is an approximately finite-dimensional C^*-algebra and $\mathcal{B}$ is a separable C^*-algebra such that $\approx_{\mathcal{B}}$ is non-smooth, then $\approx_{\mathcal{A}}$ is Borel reducible to $\approx_{\mathcal{B}}$.

Corollary (Elliot 1977)

If $\mathcal{A}$, $\mathcal{B}$ are approximately finite-dimensional C^*-algebras such that $\approx_{\mathcal{A}}$, $\approx_{\mathcal{B}}$ are non-smooth, then $\approx_{\mathcal{A}}$ and $\approx_{\mathcal{B}}$ are Borel bireducible.

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If G, H are countable locally finite groups, neither of which is abelian-by-finite, then $\approx_G$ and $\approx_H$ are Borel bireducible.

Even less as expected ...

Theorem (Sutherland 1983)

Let $H = \bigoplus_{n \in \mathbb{N}} \mathrm{Sym}(3)$. If G is any countable amenable group, then $\approx_G$ is Borel reducible to $\approx_H$.

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Remark

The theorem ultimately depends upon the Ornstein-Weiss Theorem that if G, H are countable amenable groups, then any free ergodic measure-preserving actions of G, H are orbit equivalent.

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- Then the unitary dual $\mathrm{Irr}_\infty(A) = C_3^{\mathbb{N}}$ is the product of countably many copies of the cyclic group

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- Let $Z = \{\, \xi, \xi^2 \,\}^{\mathbb{N}} \subseteq C_3^{\mathbb{N}}$ and let μ be the usual product probability measure on Z.
- If $\mathcal{H}$ is any Hilbert space, then we can define a representation π of A on $L^2(Z, \mathcal{H})$ by

$$(\pi(a) \cdot f)(z) = z(a)\, f(z).$$

Some representations of $H = \bigoplus_{n \in \mathbb{N}} \mathrm{Sym}(3)$

- Now we extend π to a representation of $H = A \rtimes K$ by defining a suitable action of K on $L^2(Z, \mathcal{H})$.

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- Now we extend π to a representation of $H = A \rtimes K$ by defining a suitable action of K on $L^2(Z, \mathcal{H})$.
- The conjugation action of K on A induces a free ergodic action of K on (Z, μ).

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- The conjugation action of K on A induces a free ergodic action of K on (Z, μ).
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- Finally we add a "twist" and define the action π_σ of K on $L^2(Z, \mathcal{H})$ by

$$(\pi_\sigma(k) \cdot f\,)(z) = \sigma(k^{-1}, z)^{-1}\, f(k^{-1} \cdot z).$$

where $\sigma : K \times Z \to U(\mathcal{H})$ is an irreducible cocycle.

Irreducible cocycles

Definition

A Borel map $\sigma : K \times Z \to U(\mathcal{H})$ *is a cocycle if for all* $g, h \in K$,

$$\sigma(gh, z) = \sigma(g, h \cdot z)\, \sigma(h, z) \qquad \mu\text{-a.e. } z \in Z.$$

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Example

If $\psi : K \to U(\mathcal{H})$ is a representation, then we can define an associated cocycle $\sigma_\psi : K \times Z \to U(\mathcal{H})$ by

$$\sigma_\psi(g, z) = \psi(g).$$

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A unitary representation $\varphi : G \to U(\mathcal{H})$ *is irreducible if and only if*

$$\{\, B \in \mathcal{L}(\mathcal{H}) \mid \varphi(g)\, B = B\, \varphi(g) \text{ for all } g \in G \,\} = \mathbb{C}\, Id_{\mathcal{H}} \,.$$

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- If $\alpha, \beta : K \times Z \to U(\mathcal{H})$ are cocycles, then $\mathrm{Hom}(\alpha, \beta)$ consists of the bounded Borel maps $b : Z \to \mathcal{L}(\mathcal{H})$ such that for all $g \in K$,

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The heart of the matter

If $K' \curvearrowright (\, Z', \mu' \,)$ is orbit equivalent to $K \curvearrowright (\, Z, \mu \,)$, then the "cocycle machinery" is isomorphic via a Borel map.

Coding representations in cocycles

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- Let G be any countable amenable group and let $\Gamma = G \times \mathbb{Z}$.
- Let $X = 2^{\Gamma}$ and let ν be the product probability measure on X.
- Then the shift action of Γ on (X, ν) is (essentially) free and strongly mixing.
- For each irreducible representation $\varphi : G \to U(\mathcal{H})$, we can define an irreducible cocycle $\sigma_{\varphi} : (G \times \mathbb{Z}) \times X \to U(\mathcal{H})$ by

$$\sigma_{\varphi}(gz, x) = \varphi(g)$$

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- If $b \in \mathrm{Hom}(\sigma_\varphi, \sigma_\varphi)$, then for all $gz \in G \times \mathbb{Z}$ and ν-a.e. $x \in X$,

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- Since the action of $G \times \mathbb{Z}$ is strongly mixing, $\mathbb{Z}$ acts ergodically on X and hence b is ν-a.e. constant.
- By Schur's Lemma, b is a scalar multiple of the identity.

Summing up ...

Definition

Let $\mathrm{Irr}_\infty(E_0)$ *be the space of irreducible cocycles*

$$\sigma : K \times Z \to U(\mathcal{H})$$

and let $\approx_{E_0}$ *be the equivalence relation defined by*

$$\sigma \approx_{E_0} \tau \quad \iff \quad \mathrm{Hom}(\sigma, \tau) \neq 0.$$

Theorem

If the countable group G *is amenable but not abelian-by-finite, then the unitary equivalence relation* $\approx_G$ *is Borel bireducible with* $\approx_{E_0}$*.*

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Remark

Here we can replace E_∞ with the universal treeable relation $E_{\infty T}$.

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Hopefully False Conjecture

If G, H are countable nonamenable groups, then $\approx_G$ *and* $\approx_H$ *are Borel bireducible.*

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The End