A set-theoretic analysis of the quotient problem and the biorthogonal system problem

Stevo Todorcevic

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Outline

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1. The dual of the Schauder basic sequence problems

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6. Mazur Intersection Property for Asplund Spaces
7. Finite-dimensional approximations

Schauder basic sequences

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A sequence $\{x_\gamma : \gamma \in \Gamma\}$ in some Banach space X indexed by a set of ordinals Γ is **basic**, or **Schauder basic**, in E if it is normalized, independent and if there is a constant $C \geq 1$ such that

$$\Big\| \sum_{i \in I} a_i x_i \Big\| \leq C \Big\| \sum_{j \in J} a_j x_j \Big\|$$

for any pair $I \sqsubseteq J$ of finite subsets of Γ such that I is an **initial segment** of J and for every sequence $(a_j : j \in J)$ of scalars.

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Theorem (Mazur 1932)

Every infinite-dimensional Banach space contains an infinite basic sequence.

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and **bounded** by a constant C if

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Theorem (Godun-Kadets 1980, Plichko 1980)

If $|\Gamma| > 2^{\aleph_0}$, *then*

$$\ell_\infty^{\aleph_0}(\Gamma) = \{x \in \ell_\infty(\Gamma) : |\mathrm{supp}(x)| \leq \aleph_0\}$$

has no fundamental biorthogonal system.

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The **cofinality** of an infinite-dimensional Banach space X is the minimal infinite cardinal θ for which there is a **increasing** sequence X_ξ $(\xi < \theta)$ of **proper closed** subspaces of X such that $\bigcup_{\xi<\theta} X_\xi$ is dense in X.

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Theorem (Folklore ?)

If an infinite-dimensional space X has cofinality ω then X has a quotient with a Schauder basis of length ω.

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We say that such an ideal $\mathcal{I}$ is a **P-ideal** if for every sequence (x_n) in $\mathcal{I}$ there is $y \in \mathcal{I}$ such that $x_n \setminus y$ is finite for all n.

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1. The ideal $[S]^{\leq \aleph_0}$ of all countable subsets of S is a P-ideal.
2. Given a family $\mathcal{F}$ of cardinality $< \mathfrak{b}$ the ideal

$$\mathcal{F}^{\perp} = \{x \in [S]^{\leq \aleph_0} : (\forall X \in \mathcal{F})|x \cap X| < \aleph_0\}$$

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Remark

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3. It is known that PID implies, for example, the Souslin Hypothesis.

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A sequence $(f_\gamma : \gamma \in \Gamma) \subseteq X^*$ is w^*-null whenever

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is finite for all $x \in X$ and $\varepsilon > 0$.

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Theorem (Josefson 1975, Nissenzweig 1975)

For every infinite-dimensional normed space X there is a normalized w^*-null sequence (f_n) $(n < \omega)$ in X^*.

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We concentrate on the alternative (2) of PID and design another P-ideal.

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3. The map $x \mapsto \sum_{\gamma \in \Gamma} f_\gamma(x) f_\gamma^*$ is a quotient map from X onto the norm-closed linear span of $\{f_\gamma^* : \gamma \in \Gamma\}$.

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An **Asplund space**, or a **strong differentiability space** is a Banach space X with the property that every continuous convex function $f : U \to \mathbb{R}$ on an open convex domain $U \subseteq X$ is Fréchet differentiable in every point of a dense G_δ-subset of U.

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sequential compactness of its dual ball with the w^*-topology,

separability of the dual of every separable subspace, etc.

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1. *Every non-separable Asplund space has an uncountable biorthogonal system.*

PID and Asplund spaces

Theorem (Todorcevic 1989)

If $\mathfrak{b} = \omega_1$ then there is a non-separable Asplund space with no uncountable biorthogonal system

Theorem (Brech-Todorcevic 2012)

Assume PID. Then every Asplund space of density $< \mathfrak{b}$ has a quotient with a Schauder basis which can be assumed to be of length ω_1 if the space is not separable.

Corollary

Assume PID. The following are equivalent:

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2. $\mathfrak{b} = \omega_2$.

PID and Asplund spaces, contd

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Remark

If M orthogonal to $\mathcal{I}$ would mean that M does not accumulate to 0^* we would be easily done, since if X is non-separable, 0^* is not a G_δ point of B_{X^*}.

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Lemma (Key)

The alternative (2) fails for the ideal $\mathcal{I} \restriction (K \cup -K) \setminus \{0^*\}$.

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A Banach space with a Fréchet differentiable norm has the MIP.

Theorem (JiménezSevilla-Moreno 1997)

Suppose that a Banach space X has a biorthogonal system $\{(x_i, f_i : i \in I)\} \subseteq X \times X^*$ such that

$$X^* = \overline{\mathrm{span}}\{f_i : i \in I\}$$

Then X admits an equivalent norm with the MIP.

Theorem (JiménezSevilla-Moreno 1997)

Suppose that a non-separable Banach space X has an equivalent norm with the MIP. Then for every $\varepsilon > 0$ there is an uncountable ε-biorthogonal system $\{(x_i, f_i : i \in I)\} \subseteq X \times X^*$, i.e., a system such that

$$f_i(x_i) = 1 \textit{ and } |f_i(x_j)| \leq \varepsilon \textit{ for } i \neq j.$$

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Does every Asplund space has an equivalent norm with the MIP?

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Remark

This is natural to ask in view of Mazur's original theorem since a Banach space with a Fréchet differentiable norm is necessarily an Asplund space.

MIP for Asplund spaces of small density

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Theorem (Bačák-Hájek 2008)

Suppose X is an Asplund space of density $\aleph_1$ with an uncountable biorthogonal theorem. Then X contains an uncountable normalizeed sequence which is w^*-null as a subset of X^{**}.

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What is the correct bound here $\mathfrak{p}$ of $\mathfrak{b}$?

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For $n < \omega$ and $\beta < \omega_1$, set

$$H_n(\beta) = \{\beta\} \cup \{\alpha < \beta : \Delta(\alpha, \beta) = n \text{ and } \rho_1(\alpha, \beta) \leq a_\beta(n)\}.$$

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Theorem (Todorcevic 1989)

$X_{\mathcal{H}}$ is a nonseparable Asplund space whose weak topology is hereditarily Lindelöf. So X has no uncountable biorthogoal systems nor equivalent norms with the MIP.

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Which properties of the cofinal family $\mathcal{F} \subseteq [\omega_1]^{<\omega}$ will give us a rich spectrum of limit spaces $X_\mathcal{F}$?

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Remark

We give an answer to this question by borrowing both from the theory of ρ-functions on ω_1 and known forcing constructions of Boolean algebras and Banach spaces.

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When n_t and the cardinality of t and r_t depend only on the rank of t in $\mathcal{F}$, we say that $\mathcal{F}$ is **homogeneous**.

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Note that condition indeed passes from rank k to rank $k+1$

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Definition

We say that $t \in \mathcal{F}$ **captures** a finite Δ-system a_ξ $(\xi \in I)$ of finite subsets of ω_1 with root a if there is a injection

$$\varphi : I \to \{1, ..., n_t\}$$

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Remark

We really want that for $\xi, \eta \in I$ the natural isomorphism between $s_{\varphi(\xi)}$ and $s_{\varphi(\eta)}$ moves a_ξ to a_η. We call this sort of capturing the **positional capturing**. We do not add this here since we will be essentially free to choose the sequence n_t to be rather fast relative to the cardinalities of the tails $s_i \setminus r_t$ of terms of the decomposition of t.

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The notions of capturing and density alow many variations. What seem easiest to use is capturing of arbitrarily large finite subsystems a_ξ $(\xi \in I)$ of an uncountable Δ-system with $\varphi : I \to \{1, ..., n_t\}$ whose range is an **initial segment** of $\{1, ..., n_t\}$ and that a_ξ gets moved to a_η in the natural isomorphism between $s_{\varphi(\xi)}$ and $s_{\varphi(\eta)}$.

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Problem

Does any of the two assumptions $\mathfrak{b} = \omega_1$ or $\mathfrak{p} = \omega_1$ give us homogeneous cofinal families of finite subsets of ω_1 that (in some sense) capture uncountable Δ-systems of finite subsets of ω_1?

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Assume $\diamondsuit$. Then there is a cofinal homogeneous family $\mathcal{F}$ of finite subsets of ω_1 which totally and positionally captures an arbitrary uncountable Δ-system of finite subsets of ω_1.

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Moreover, the corresponding sequences of cardinalities $(r_k)_k$, $(m_k)_k$ and $(n_k)_k$ can essentially be given in advance.

Remark

Only the sequence $(r_k)_k$ of cardinalities of roots needs to satisfy certain mild requirement, the other two sequences are arbitrary, modulo the requirements

$$m_0 = 1, r_{k+1} < m_k, n_{k+1} > 1, \text{ and } m_{k+1} = r_{k+1} + n_{k+1}(m_k - r_{k+1}).$$

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Proposition

If $\mathcal{F}$ is a homogeneous cofinal family of finite subsets of ω_1 which positionally and totally captures uncountable Δ-systems then $X_\mathcal{F}$ is a nonseparable Asplund space with no uncountable biorthogonal system.