The minimal flows of S_∞

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Minimal flows

G — topological group.

A G-flow, $G \curvearrowright X$, is a compact Hausdorff space X equipped with a continuous action of G.

Morphisms: if X and Y are G-flows, a homomorphism from X to Y is a continuous map $\pi \colon X \to Y$ that commutes with the G-actions, i.e.

$$\pi(g \cdot x) = g \cdot \pi(x) \quad \text{for all } x \in X, g \in G.$$

A flow is minimal if it has no proper subflows or, equivalently, if every orbit is dense.

Compactness + Zorn's lemma $\implies$ every flow contains a minimal subflow.

Minimal flows are some of the main objects of study in topological dynamics.

The universal minimal flow

For every group G, there exists a universal minimal G-flow (a minimal G-flow that maps onto any other minimal G-flow). For example, one can take any minimal subflow of the product

$$\prod\{M : M \text{ is a minimal } G\text{-flow}\}.$$

It is a standard theorem that the universal property characterizes the universal minimal flow up to isomorphism.

Alternatively, if G is discrete, the pointed flow $(\beta G, 1_G)$ is a universal pointed flow for G, i.e. for every flow $G \curvearrowright X$ and every $x_0 \in X$, there exists a homomorphism

$$\pi : \beta G \to X \quad \text{such that} \quad \pi(1_G) = x_0.$$

Consequently, any minimal subflow of βG is universal for all the minimal flows.

The universal minimal flow (cont.)

We see that in this case, the universal minimal flow is non-metrizable and not amenable to a concrete description.

This is reflected by the large variety of minimal flows that exist for discrete groups. For example, studying (minimal) subflows of the shift $\mathbf{Z} \curvearrowright 2^{\mathbf{Z}}$ is a subject of its own (symbolic dynamics).

If G is compact, then every minimal G-flow is transitive, that is, of the form $G \curvearrowright G/H$, where H is a closed subgroup of G and the universal minimal flow is the left action of G on itself.

Another case in which the situation trivializes is when the group G is extremely amenable, i.e. its universal minimal flow is a singleton. This turns out to be the case for many symmetry groups of continuous objects (the infinite-dimensional unitary group, the group of measure-preserving transformations of the interval, etc.)

The universal minimal flow of S_∞

S_∞ is the group of all permutations of the natural numbers, equipped with the pointwise convergence topology: a basis at the identity is given by the open subgroups

$$V_n = \{g \in S_\infty : g \cdot i = i \text{ for all } i = 0, \dots, n-1\}.$$

The set

$$\mathrm{LO} = \{x \in 2^{\mathbf{N}\times\mathbf{N}} : x \text{ is a linear order}\}$$

is a compact subset of $2^{\mathbf{N}\times\mathbf{N}}$ on which S_∞ acts via the logic action:

$$a <_{g\cdot x} b \iff g^{-1} \cdot a <_x g^{-1} \cdot b, \quad g \in S_\infty, x \in \mathrm{LO}, a, b \in \mathbf{N}.$$

The action is minimal: if $U \subseteq \mathrm{LO}$ is the open set $0 < 1 < 2 < 3$ and $x \in \mathrm{LO}$ is such that $3 <_x 2 <_x 0 <_x 1$, then there is an obvious permutation g that sends x in U.

The universal minimal flow of S_∞ (cont.)

Theorem (Glasner–Weiss)

The flow $S_\infty \curvearrowright \mathrm{LO}$ is the universal minimal flow of S_∞.

Let $\eta_0 \in \mathrm{LO}$ be a linear order isomorphic to $(\mathbf{Q}, <)$. For $x \in \mathrm{LO}$, $x \in S_\infty \cdot \eta_0$ iff x is isomorphic to η_0 iff

$$\forall a, b\ \exists c \quad a <_x c <_x b; \quad \text{and}$$
$$\forall a\ \exists b, c \quad b <_x a <_x c,$$

which shows that the orbit $S_\infty \cdot \eta_0$ is G_δ.

Let H be the stabilizer of η_0 in S_∞. Then H is isomorphic to $\mathrm{Aut}(\mathbf{Q}, <)$ and

Theorem (Pestov)

The group $\mathrm{Aut}(\mathbf{Q}, <)$ is extremely amenable.

The universal minimal flow of S_∞ (proof)

Say that the homogeneous space G/H is precompact if the natural uniformity, whose entourages of the diagonal are

$$\mathcal{U}_V = \{(gH, vgH) : v \in V, g \in G\}, \quad V \text{ is a symmetric nbhd of } 1_G,$$

is precompact. Equivalently, for every V, there exists a finite F such that $VFH = G$.

H — the stabilizer of η_0 in LO. S_∞/H is precompact. It is not difficult to check that

$$(S_\infty \curvearrowright \widehat{S_\infty/H}) \cong (S_\infty \curvearrowright \mathrm{LO}).$$

Let now $S_\infty \curvearrowright X$ be any flow. As H is extremely amenable, there exists $x_0 \in X$ fixed by H. Define a map $\phi: S_\infty/H \to X$ by $\phi(gH) = g \cdot x_0$. ϕ is uniformly continuous and extends to a map $\hat{\phi}: \mathrm{LO} \to X$.

Invariant closed equivalence relations

As LO is the universal minimal flow for S_∞, for any other minimal flow X, there exists a quotient S_∞-map $\pi\colon \mathrm{LO} \to X$. To every such map corresponds an equivalence relation $\mathcal{R}_\pi$ on LO defined by

$$x \,\mathcal{R}_\pi\, y \iff \pi(x) = \pi(y).$$

$\mathcal{R}_\pi$ is an invariant, closed equivalence relation, icer for short. Conversely, any icer gives a quotient of LO.

Classifying the minimal flows of S_∞ therefore boils down to classifying the icers on LO.

It turns out that there are only countably many such icers, each one of them can be generated by a single pair $(x, y) \in \mathrm{LO} \times \mathrm{LO}$, and each quotient can be expressed as the set of models of a universal theory (like the theory of linear orders) and is, in particular, zero-dimensional.

The quotients coming from groups

As LO has a G_δ orbit (that of η_0), it is easy to see that any other minimal S_∞-flow X must also have a G_δ orbit and that the quotient map $\pi\colon \mathrm{LO} \to X$ sends η_0 in that orbit.

Let $x_0 = \pi(\eta_0)$ and let H_{x_0} be the stabilizer of x_0. Then $H \le H_{x_0} \le S_\infty$.

Conversely, any such subgroup H' is co-precompact and there exists a natural quotient map

$$\mathrm{LO} \cong \widehat{S_\infty/H} \to \widehat{S_\infty/H'}.$$

So for every minimal flow X, we have the factorization

$$\mathrm{LO} \to \widehat{S_\infty/H_{x_0}} \to X,$$

where the second map is almost one-to-one (one-to-one on a comeager set).

Quotients coming from groups (cont.)

We define certain S_∞-maps $\pi\colon \mathrm{LO} \to 2^{\mathbf{N}^k}$. Then $\pi(\mathrm{LO})$ is a minimal flow of S_∞.

- the betweenness relation (BR) ($k = 3$)

$$B_x(a, b, c) \iff (a <_x b <_x c) \vee (c <_x b <_x a);$$

- the circular order (CO) ($k = 3$)

$$K_x(a, b, c) \iff (a <_x b <_x c) \vee (b <_x c <_x a) \vee (c <_x a <_x b);$$

- the separation relation (SR) ($k = 3$)

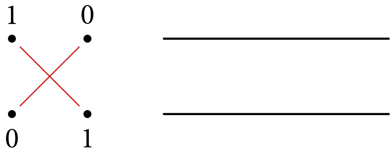
$$S_x(a, b, c, d) \iff \big(K_x(a, b, c) \wedge K_x(b, c, d) \wedge K_x(c, d, a)\big) \vee \big(K_x(d, c, b) \wedge K_x(c, b, a) \wedge K_x(b, a, d)\big)$$

The rest

- $\mathrm{LO}_{m,n}$ $(k = m + n + 1)$

$$P^x_{m,n}(a_1, \ldots, a_m, b, c_1, \ldots, c_n) \iff (\bar{a} <_x b <_x \bar{c}) \wedge \Big(\bigwedge_{i \neq j} a_i \neq a_j\Big) \wedge \Big(\bigwedge_{i \neq j} c_i \neq c_j\Big).$$

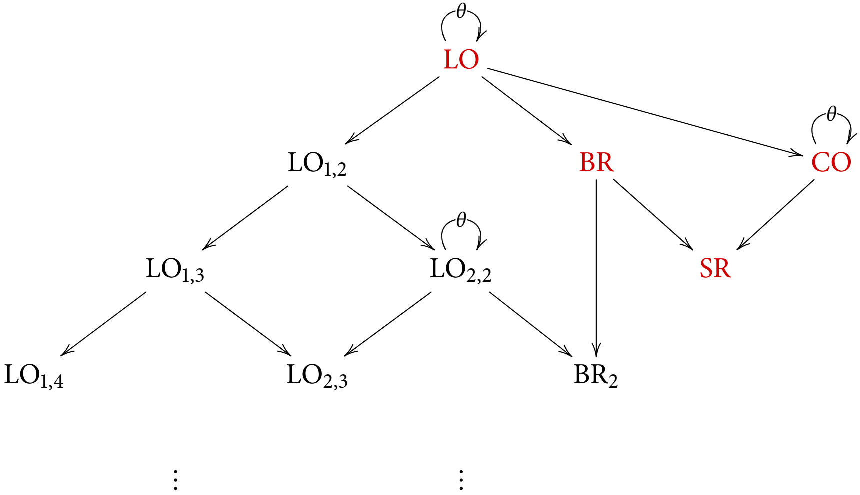
Two linear orders which are identified in $\mathrm{LO}_{2,1}$:



- $\mathrm{BR}_n = \mathrm{LO}_{n,n}/\text{flip}$ $(k = 2n + 1)$

$$Q^x_n(a_1, \ldots, a_n, b, c_1, \ldots, c_n) \iff P^x_{n,n}(\bar{a}, b, \bar{c}) \vee P^x_{n,n}(\bar{c}, b, \bar{a}).$$

The complete picture



The arrows represent all possible homomorphisms between the flows.

History

- Frasnay in 1965 classifies certain sequences of finite groups related to “bi-orders” (sets carrying two linear orders), a classification that basically amounts to the picture above;
- Cameron in 1976 (unaware of the work of Frasnay) classifies all groups between $\mathrm{Aut}(\mathbf{Q}, <)$ and S_∞ (the red nodes of the diagram);
- Hodges, Lachlan and Shelah in 1977 independently prove a theorem about indiscernibles that also amounts to a special case of Frasnay’s work.

Some questions

- What about other automorphism groups?
- What are some conditions that will ensure that a group of automorphisms has only countably many minimal flows?
- …that all minimal flows are zero-dimensional? (An example to be kept in mind is the automorphism group of the random circular order (the G_δ orbit of $\mathcal{CO}$). This group acts minimally on the circle with two orbits.)
- …that all minimal flows are realized as the models of theories in a finite language?