Towards a structure theory of Maharam algebras

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Outline

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Introduction

Definition

$\mathcal{B}$ is a ***measure algebra*** *if there exists a measure $\mu : \mathcal{B} \to [0,1]$ which is σ-additive, strictly positive and such that $\mu(\mathbf{1}_\mathcal{B}) = 1$.*

Proposition

Let $\mathcal{B}$ be a measure algebra. Then

1. *$\mathcal{B}$ satisfies* ***the countable chain condition****, i.e. if $\mathcal{A} \subseteq \mathcal{B} \setminus \{\mathbf{0}\}$ is such that $a \wedge b = \mathbf{0}$, for all $a, b \in \mathcal{A}$ such that $a \neq b$ then $\mathcal{A}$ is at most countable.*
2. *$\mathcal{B}$ is* ***weakly distributive****, i.e. if $\{b_{n,k}\}_{n,k}$ is a double sequence of elements of $\mathcal{B}$ then*

$$\bigwedge_n \bigvee_k b_{n,k} = \bigvee_{f:\mathbb{N}\to\mathbb{N}} \bigwedge_n \bigvee_{i<f(n)} b_{n,i}$$

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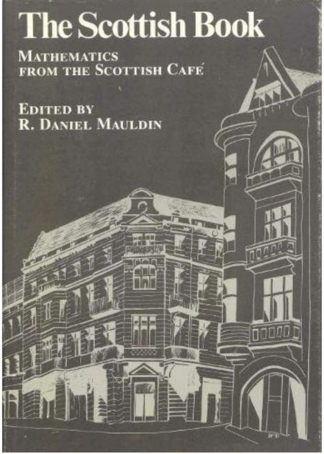

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Let $\mathcal{B}$ be a complete Boolean algebra satisfying 1. and 2. Is $\mathcal{B}$ a measure algebra?

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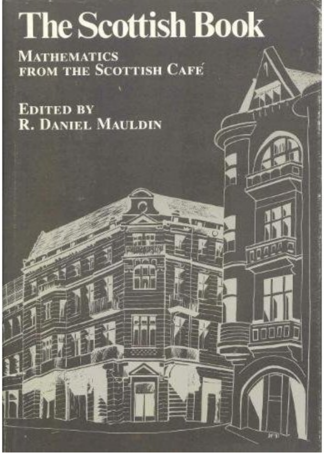

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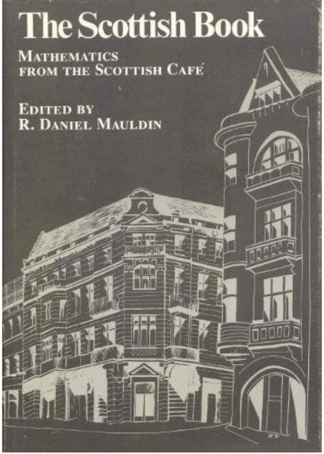

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Definition (Maharam)

We say that a sequence $\{x_n\}_n$ *of elements of* $\mathcal{B}$ ***strongly converges*** *to* x *and we write* $x_n \to x$ *if*

$$\limsup_n x_n = \liminf_n x_n = x$$

Let $X \subseteq \mathcal{B}$. We define:

$$\overline{X} = \{x \in \mathcal{B} : \text{ there exists } \{x_n\}_n \subseteq \mathcal{B} \text{ such that } x_n \to x\}.$$

Proposition (Maharam)

1. *If* $\mathcal{B}$ *satisfies Von Neumann's conditions that the strong convergence defines a topology on* $\mathcal{B}$.
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Question

Let $\mathcal{B}$ be a complete Boolean algebra verifying the conditions of Von Neumann. When is the strong topology on $\mathcal{B}$ metrizable?

Definition (Maharam)

A ***continuous submeasure*** *on $\mathcal{B}$ is a function $\mu : \mathcal{B} \to [0, 1]$ such that*

1. $\mu(x) = 0$ *iff* $x = \mathbf{0}$
2. *If* $x \leq y$ *then* $\mu(x) \leq \mu(y)$
3. $\mu(x \vee y) \leq \mu(x) + \mu(y)$
4. *If* $x_n \to x$ *then* $\mu(x_n) \to \mu(x)$.

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Let $\mathcal{B}$ be a complete Boolean algebra. Then the strong topology on $\mathcal{B}$ is metrizable iff $\mathcal{B}$ admits a continuous submeasure.

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Therefore, Von Neumann's problem decomposes into two questions.

Question (1)

If $\mathcal{B}$ is a c.c.c. weakly distributive complete Boolean algebra is $\mathcal{B}$ a Maharam algebra?

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Is every Maharam algebra a measure algebra?

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Relative consistency results

We now have a fairly complete answer to Question 1.

Theorem (Farah, V.)

Suppose every c.c.c. weakly distributive complete Boolean algebra is a Maharam algebra. Then there is an inner model of ZFC with with (hyper) measurable cardinals.

Theorem (Balcar, Jech, Pazak, V.)

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1. *We say that μ is* ***exhaustive*** *if for every sequence $\{a_n\}_n$ of pairwise disjoint elements of $\mathcal{B}$ we have $\lim_n \mu(a_n) = 0$.*
2. *We say that μ is* ***uniformly exhaustive*** *if for every $\epsilon > 0$ there exists n such that there are no n pairwise disjoint elements $a_1, \ldots, a_n$ of $\mathcal{B}$ such that $\mu(a_i) \geq \epsilon$, for all i.*

Theorem (Kalton & Roberts, 1983)

If a submeasure μ is uniformly exhaustive then it is equivalent to a measure.

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Let $\mathcal{A} \subseteq \mathcal{P}(X)$ be a Boolean algebra and $\nu : \mathcal{A} \to [0,1]$ a positive submeasure on $\mathcal{A}$. We say that μ is ***pathological*** *if for every $\epsilon > 0$ there is a finite sequence $(b_i)_{i \leq n}$ of elements of $\mathcal{A}$ such that $\nu(b_i^c) \leq \epsilon$, for all i, and for all $x \in X$*

$$|\{i : x \in b_i\}| \leq \epsilon n.$$

If $\mathcal{A} \subset \mathcal{P}(X)$ is a Boolean algebra, ν a pathological submeasure and μ a measure. Then ν et μ are orthogonal, i.e., for all $\epsilon > 0$ there is $b \in \mathcal{A}$ such that $\nu(b^c) \leq \epsilon$ and $\mu(b) \leq \epsilon$.

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Theorem (Talagrand, 2005)

Let $T = \prod_n 2^n$. Let $\mathcal{A}$ be the algebra of clopen subsets of T. Then there is an exhaustive pathological submeasure ν on $\mathcal{A}$.

Once we have such a submeasure ν we can use the usual construction of the Lebesgue measure to extend it to all Borel subsets of T. In this way, we obtain a continuous submeasure $\bar{\nu}$ on $Bor(T)$. Let $\mathcal{I}_{\bar{\nu}}$ be the ideal of null sets in the sense of $\bar{\nu}$. Then $\mathcal{B} = Bor(T)/\mathcal{I}_{\bar{\nu}}$ is a Maharam algebra which is not a measure algebra.

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There exists a Maharam algebra which is not a measure algebra.

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Structure of Maharam algebras

Little is known about the properties of Maharam algebras, in particular the one constructed by Talagrand. First we show that they share some properties of measure algebras.

Definition

*Let $\mathcal{B}$ be a Boolean algebra. A sequence $\{b_n\}_n$ of elements of $\mathcal{B}$ is **splitting** if for every infinite $I \subseteq \omega$ and $\alpha \in \{0,1\}^I$ $\bigwedge_{n \in I} b_n^{\alpha(n)} = \mathbf{0}$ where $b^0 = b$ and $b^1 = \mathbf{1} - b$.*

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Let $\mathcal{B}$ be a non atomic Maharam algebra. Then the Cohen algebra (of regular open subsets of $\mathcal{C}$) can be embedded into $\mathcal{B} \times \mathcal{B}$.

In the case of measure algebras there is a nice classification result. Given an infinite cardinal κ let λ_κ be the usual product measure on $\{0,1\}^\kappa$. Let $\mathcal{B}_\kappa$ be the σ-algebra of Baire sets in $\{0,1\}^\kappa$ and $\mathcal{N}_\kappa$ the ideal of λ_κ-null sets. Finally, we let $\mathcal{M}_\kappa$ be the algebra $\mathcal{B}_\kappa/\mathcal{N}_\kappa$. $\mathcal{M}_\kappa$ is the homogeneous measure algebra of density κ.

Theorem (Maharam)

For every non atomic measure algebra $\mathcal{M}$ there is a countable set I of cardinals such that

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For Maharam algebras no such simple classification is possible. First, we define a notion of rank.

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Let $\mathcal{B}$ be a Boolean algebra, ν an exhaustive submeasure on $\mathcal{B}$ and $\epsilon > 0$. Let $\mathcal{D}_\epsilon(\mathcal{B})$ be the set of all finite pairwise disjoint subsets F of $\mathcal{B}$ such that $\nu(a) \geq \epsilon$, for all $a \in F$. Define the order on $\mathcal{D}_\epsilon(\mathcal{B})$ by $F \leq G$ iff $F \subseteq G$.

Since ν is exhaustive it follows that $\mathcal{D}_\epsilon(\mathcal{B})$ is well founded. Let $\mathrm{rk}_\epsilon(\nu)$ be the rank of this ordering. Finally, let

$$\mathrm{rk}(\nu) = \sup\{\mathrm{rk}_\epsilon(\nu) : \epsilon > 0\}.$$

Fact

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Fact

If ν is an exhaustive submeasure which is not uniformly exhaustive then $\mathrm{rk}(\nu) \geq \omega^\omega$.

Question

What is the rank of Talagrand's submeasure?

Proposition (Fremlin)

Let ν be the pathological exhaustive submeasure constructed by Talagrand. Then $\omega^\omega \leq \mathrm{rk}(\nu) \leq \omega^{\omega^2}$.

If ν is an exhaustive submeasure on a countable Boolean algebra $\mathcal{A}$ then $\mathrm{rk}(\nu)$ is a countable ordinal. Can we get arbitrary high countable ordinals?

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Theorem (Perovic, V.)

There exist exhaustive submeasures on the Boolean algebra $\mathcal{A}$ of clopen subsets of T of arbitrary high rank below ω_1.

Corollary

The set of exhaustive submeasure on $\mathcal{A}$ is a true Π^1_1 set, i.e. it is not Borel.

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Definition (Schreier families)

For every countable ordinal α, we define a family $\mathcal{S}_\alpha$ of finite subsets of $\mathbb{N}$ as follows.

1. $\mathcal{S}_0 = \{\{n\} : n \in \mathbb{N}\} \cup \{\varnothing\}$.
2. *Given $\mathcal{S}_\alpha$ we let*

$$\mathcal{S}_{\alpha+1} = \{\bigcup_{i<n} F_i : n \leq F_0 < F_1 < \ldots < F_{n-1}, F_i \in \mathcal{S}_\alpha (i < n)\}.$$

3. *If α is a limit ordinal, fix an increasing sequence $(\alpha_n)_n$ converging to α and let*

$$S_\alpha = \bigcup_n \{F \in \mathcal{S}_{\alpha_n} : n \leq F\} \cup \{\varnothing\}.$$

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Definition

*Fix a countable ordinal α. We say that F is **maximal** for $\mathcal{S}_\alpha$ if $F \in \mathcal{S}_\alpha$ and whenever $G \in \mathcal{S}_\alpha$ is such that $F \subseteq G$ then $G = F$.*

Definition

Fix an ordinal α. For every finite subset F of $\mathbb{N}$ we define $m_i^\alpha(F)$ by induction on i as follows.

1. $m_0^\alpha(F) = \min(F)$.
2. *Suppose $m_i^\alpha(F)$ has been defined. Let $m_{i+1}^\alpha(F)$ be the least $m \in F$ (if it exists) such that $F \cap [m_i^\alpha(F), m)$ is $\mathcal{S}_\alpha$ maximal.*

*Let $k_\alpha(F)$ be the least k such that $m_k^\alpha(F)$ is not defined. We set $F^* = \{m_i^\alpha(F) : i < k_\alpha(F)\}$. Elements of F^* are called the **leaders** of F. Finally, set $\|F\|_\alpha = k_\alpha(F)$.*

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Proposition

Fix $\alpha < \omega_1$ and finite $A, B \subseteq \mathbb{N}$.

1. *If $A \subseteq B$ then $||A||_\alpha \leq ||B||_\alpha$.*
2. *if $A = \{a_0 < \ldots < a_{n-1}\}$ and $B = \{b_0 < \ldots < b_{n-1}\}$ with $a_i \leq b_i$, for $i < n$, then $||A||_\alpha \leq ||B||_\alpha$.*
3. $||A \cup B||_\alpha \leq ||A||_\alpha + ||B||_\alpha$.

We call $|| \cdot ||_\alpha$ the **α-Schreier norm**.

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Recall that $T = \prod_n 2^n$. Let P be the collection of all finite partial functions s such that $\text{dom}(s) \subseteq \mathbb{N}$ and $s(k) < 2^k$, for all $k \in \text{dom}(s)$. For $s \in P$ Let

$$N(s) = \{f \in T : s \subseteq T\}.$$

Then $N(s)$ is a typical clopen subset of T. We adapt Talagrand's construction to show the following.

Theorem (Perovic, V.)

Suppose α is a countable ordinal. Then there is an exhaustive submeasure ν_α on clopen subsets of T such that

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Definition

1. *Let* $P_\alpha = \{s \in P : \|dom(s)\|_\alpha \leq 1\}$.
2. *Let* $\mathcal{D}_\alpha$ *be the collection of all finite subsets* F *of* P_α *such that* $s \perp t$*, for all* $s, t \in F$ *such that* $s \neq t$*. We let* $G \leq F$ *if* $F \subseteq G$*.*

Proposition

$\mathcal{D}_\alpha$ *is well founded and* $\mathrm{rk}(\mathcal{D}_\alpha) \geq \omega^\alpha$.

Sketch of proof : The fact that $\mathcal{D}_\alpha$ is well founded follows from a straightforward application of the Δ-system lemma. We show that $\mathrm{rk}(\mathcal{D}_\alpha)$ is at least ω^α by induction on α. We consider the following game.

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The game G_α:

I	ρ_0		ρ_1		ρ_2		$\dots$	
II		s_0		s_1		s_2		$\dots$

Player I plays a decreasing sequence of ordinals smaller than ω^α and Player II plays pairwise incompatible elements of P_α. The game has to stop after finitely many moves. Player II wins the game if he can continue playing till Player I reaches 0.

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Given $s \in P$ and an integer n we define the ***shift*** $\mathrm{sh}_n(s)$ *of s by n. We let $dom(\mathrm{sh}_n) = \{k + n : k \in dom(s)\}$ and $\mathrm{sh}_n(k + n) = s(k)$, for all $k \in dom(s)$.*

We show that Player II has a winning strategy in G_α by induction on α. Suppose first $\alpha = \beta + 1$ and fix a winning strategy σ_β for II in G_β. Let ρ_0 be the first move of Player I in G_α. We may assume ρ_0 is of the form $\omega^\beta \cdot n_0 + \xi_0$, for some integer n_0 and $\xi_0 < \omega^\beta$. We pick an integer $a_0 \geq 2$ such that $2^{a_0} > n_0$. Let t_0 be the response of σ_β in the game G_β if Player I plays as his first move ξ_0. In G_α we play $s_0 = \{(a_0, n_0)\} \cup \mathrm{sh}_{a_0+1}(t_0)$. It is easy to check that $s_0 \in S_\alpha$. As long as Player I plays ordinals ρ_i of the form $\omega^\beta \cdot n_0 + \xi_i$, for some $\xi_i < \omega^\beta$, Player II uses the strategy σ_β to obtain t_i and then plays $s_i = \{(a_0, n_0)\} \cup \mathrm{sh}_{a_0+1}(t_i)$.

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Suppose at some stage i Player I plays an ordinal ρ_i of the form $\omega^\beta \cdot n_1 + \xi_i$, for some $n_1 < n_0$. Then Player II starts another round of the game G_β. Let t_i be the response of σ_β if Player I plays ξ_i as the first move. Then Player II plays $s_i = \{(a_0, n_1)\} \cup \mathrm{sh}_{a_0+1}(t_i)$. Then Player II keeps simulating the game G_β as long as Player I plays ordinals of the form $\omega^\beta \cdot n_1 + \xi$, shifting the response of σ_β by $a_0 + 1$ and adding $\{(a_0, n_1)\}$, etc. In this way Player II produces pairwise incompatible elements of P. Since the union of $\{a_0\}$ and an element of S_β is in S_α it follows that all these partial functions are in P_α. This completes the proof in the successor case.

The limit case is similar.

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