Transfinite constructions in $V = L$

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2-point sets

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Theorem. (Bouhjar, Dijkstra, and van Mill) It cannot be F_σ!

Inductive proof

Standard proof of the existence:

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Question

The set of possible choices is very large. Could it be done in a "nice" way?

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Method

Miller's method is frequently needed. The proof uses effective descriptive set theory and model theory.

General method

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Cofinality in the Turing degrees

Definition. A set $X \subset \mathbb{R}$ is *cofinal in the Turing degrees* if $(\forall z \in \mathbb{R})(\exists y \in X)(y \leq_T x)$.

Compatibility

Definition. Let $F \subset \mathbb{R}^{\leq\omega} \times \mathbb{R} \times \mathbb{R}$, and $X \subset \mathbb{R}$. We say that X *is compatible with* F if there exist enumerations $\mathbb{R} = \{p_\alpha : \alpha < \omega_1\}$, $X = \{x_\alpha : \alpha < \omega_1\}$ and for every $\alpha < \omega_1$ a sequence $A_\alpha \in \mathbb{R}^{\leq\omega}$ that is an enumeration of $\{x_\beta : \beta < \alpha\}$ in type $\leq \omega$ such that $(\forall \alpha < \omega_1)(x_\alpha \in F_{(A_\alpha, p_\alpha)})$ holds.

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Effectiveness

$\Sigma^0_1(y)$, $\Pi^0_1(y)$

Definition. Let $\{I_n : n \in \omega\}$ be a recursive enumeration of the open intervals with rational endpoints. An open set G is called *recursive in* y, iff there exists $\{n_k : k \in \omega\}$ (as an element of 2^ω) $\leq_T y$ such that $G = \cup_k I_{n_k}$. (denoted by $\Sigma^0_1(y)$).

$$\Pi^0_1(y) = \{G^c : G \in \Sigma^0_1(y)\}$$

We can define these classes similarly for subsets of ω, $\omega \times \mathbb{R}$, $\mathbb{R}^2$ etc. using a recursive enumeration of $\{n\}$, $\{n\} \times I_m$, $I_n \times I_m$ etc.

The lightface classes

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$$\Sigma^0_n(y) = \{proj_{\mathbb{R}}(A) : A \subset \mathbb{R} \times \omega, A \in \Pi^0_{n-1}(y)\},$$

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Lightface and boldface

$\boldsymbol{\Sigma}^{\mathbf{i}}_{\mathbf{j}} = \cup_{y \in \mathbb{R}} \Sigma^i_j(y)$

Strengthenings

Cofinality in the hyperdegrees

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Stronger version

Theorem 2. (V=L) Let $t \in \mathbb{R}$, $F \subset \mathbb{R}^{\leq\omega} \times \mathbb{R} \times \mathbb{R}$ be a $\Pi^1_1(t)$ set. Assume that for every $(A, p) \in \mathbb{R}^{\leq\omega} \times \mathbb{R}$ the section $F_{(A,p)}$ is cofinal in the hyperdegrees.

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Remark

The previous theorem holds true replacing $\mathbb{R}$ with $\mathbb{R}^n$, ω^ω or 2^ω.

Application

Miller's results

Theorem 2. $\Longrightarrow$ Miller's results: consistent existence of Π^1_1 MAD family, 2-point set and Hamel basis.

2-point set

Recall Theorem 1.

(V=L) If F is $\mathbf{\Pi}^1_1$ and every section $F_{(A,p)}$ is cofinal in the Turing degrees then there exists a $\mathbf{\Pi}^1_1$ set X and enumerations $\mathbb{R} = \{p_\alpha : \alpha < \omega_1\}$, $X = \{x_\alpha : \alpha < \omega_1\}$, A_α of $\{x_\beta : \beta < \alpha\}$, such that $(\forall \alpha < \omega_1)(x_\alpha \in F_{(A_\alpha, p_\alpha)})$.

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Theorem 1. $\implies$ X is a $\mathbf{\Pi}^1_1$ set.

Consequences: C^1 curves

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Consistent nonexistence

Theorem (J. Hart, K. Kunen) (PFA) For every uncountable $X \subset \mathbb{R}^2$ there exists a C^1 curve intersecting it in uncountably many points.

Consequences: C^1 curves

General version

Theorem 3. (V=L) Suppose that $G \subset \mathbb{R} \times \mathbb{R}^n$ is a Borel set and for every countable $A \subset \mathbb{R}$ the complement of the set $\cup_{p \in A} G_p$ is cofinal in the Turing degrees. Then there exists an uncountable $\mathbf{\Pi}^1_1$ set $X \subset \mathbb{R}^n$ which intersects every G_p in a countable set.

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Theorem 3. $\implies$ that under (V=L) there exists an uncountable $\mathbf{\Pi}^1_1$ set $X \subset \mathbb{R}^2$ intersecting every C^1 curve in countably many points.

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Remark

In almost every cases there are no $\mathbf{\Sigma}^1_1$ sets.

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