Idiosynchromatic Poetry

03E02, 03E10, 05C15, 05C20, 05C63

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Trends in set theory, Sunday, 8^{th} of July 2012, 11:45-12:00

Definition

$\alpha \to (\beta, \gamma)$ means
$\forall E \subset [\alpha]^2 (\exists X \in [\alpha]^\beta : [X]^2 \subset E \vee \exists X \in [\alpha]^\gamma : E \cap [X]^2 = \emptyset)$.

Remark

Here we are always referring to the order-type, i.e. $[\gamma]^\delta$ *is the set of all subsets of* γ *whose order-type is* δ*.*

Fact (AC)

For any linear order φ *we have* $\varphi \not\to (\overline{\overline{\varphi}} + 1, \omega)$ *.*

Definition

$r(\beta, \gamma)$ is the least ordinal α such that $\alpha \to (\beta, \gamma)$.

Theorem (Specker, 1957)
$r(\omega^2, m) = \omega^2$ *for all* $m < \omega$.

Theorem (Specker, 1957)
$r(\omega^l, 3) > \omega^l$ *for all* $l \in \omega \setminus 3$.

Theorem (Baumgartner, 1989)
$\mathrm{MA}_{\aleph_1}$ *implies* $r(\omega_1\omega, m) = \omega_1\omega$ *for all* $m < \omega$

Theorem (Baumgartner, 1989)
$\mathrm{MA}_{\aleph_1}$ *implies* $r(\omega_1\omega^2, m) = \omega_1\omega^2$ *for all* $m < \omega$.

Notation

$r(I_l, L_m)$ is the least n such that any digraph on n vertices contains an independent set of size l or a transitive induced subtournament of size m.

Theorem (Erdős & Rado, 1956)

$r(\omega l, m) = \omega r(I_l, L_m)$.

Theorem (Baumgartner, 1974)

$r(\lambda l, m) = \lambda r(I_l, L_m)$ for all infinite cardinals λ.

Theorem (Larson, Mitchell, 1997)

$\forall n \in \omega \setminus 2 : r(I_n, L_3) \leqslant n^2.$

Theorem (Erdős, Moser, 1964)

$\forall n \in \omega \setminus 3 : r(I_2, L_n) \leqslant 2^{n-1}.$

Theorem (Larson, Mitchell, 1997)

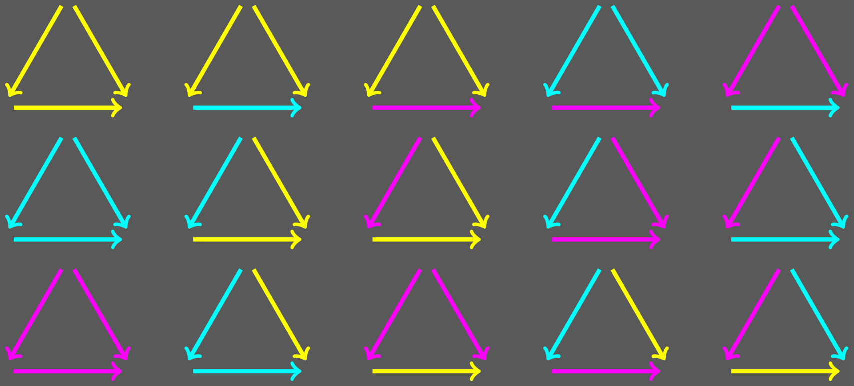
$\forall m \in \omega \setminus 3, n \in \omega \setminus 4 : r(I_2, L_n) \leqslant u(m, n)$ *with*

$$u(m,n) := \frac{1}{2}\left(2^{n-3}\left(4\binom{m+n-4}{n-1} + 6\binom{m+n-5}{n-2} + 9\binom{m+n-6}{n-3}\right) + 2^{n-4}\cdot 17\binom{m+n-6}{m-2} - 1\right)$$

	3	4	5	6	7	8	9	m
3	6	9	14	18	23	28	36	
4	9	18	25					
λ	λ	λ	λ	λ	λ	λ	λ	λ
$\lambda 2$	$\lambda 4$	$\lambda 8$	$\lambda 14$	$\lambda 28$				
$\lambda 3$	$\lambda 9$							
ω^2	ω^2	ω^2	ω^2	ω^2	ω^2	ω^2	ω^2	ω^2
$\omega^2 2$								
ω^3	ω^4	ω^4	ω^5	ω^5	ω^5	ω^5	ω^6	$\omega^{2+\lceil \mathrm{ld}(m) \rceil}$
ω^4	ω^7	ω^7	ω^{10}	ω^{10}	ω^{10}	ω^{10}		
ω^{5+l}	ω^{9+2l}	ω^{9+2l}	ω^{13+3l}	ω^{13+3l}	ω^{13+3l}	ω^{13+3l}	ω^{17+4l}	$\omega^{1+(4+l)\lceil \mathrm{ld}(m) \rceil}$
ω^ω	ω^ω	ω^ω	ω^ω	ω^ω	ω^ω	ω^ω	ω^ω	ω^ω
ω^{ω^2}	ω^{ω^2}	ω^{ω^2}						
$\kappa\lambda 2$								
$\kappa\lambda 3$								

Definition

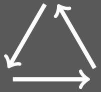
A triple is called *agreeable* if and only if it is one of the following.



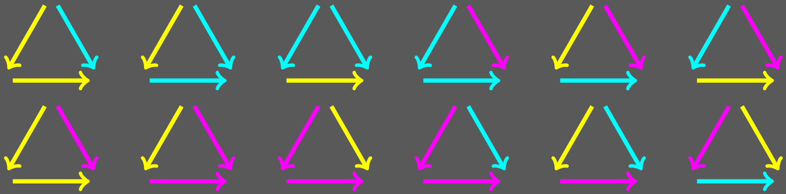
Fact

A triple is disagreeable if and only if it is either...

- *...a cyclic triple, regardless of the colouring, i.e.*



- *...or one of the following transitive triples:*



Notation

$r(I_l, A_m)$ is the least n such that any arc-3-coloured digraph on n vertices contains an independent set of size l or an induced subtournament of size m all induced 3-person-subtournaments of which are agreeable.

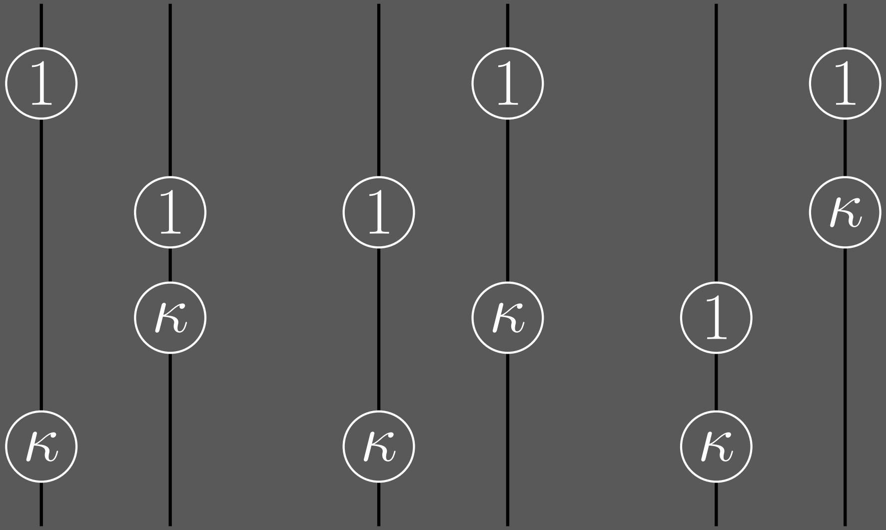
Theorem (W.)

Let κ be such that $r(\kappa, \kappa) = \kappa$ and $\kappa \in \Omega \setminus 3$—this means that $\kappa = \omega$ or κ is weakly compact.
Then $r(\kappa^2 m, n) = \kappa^2 r(I_m, A_n)$.

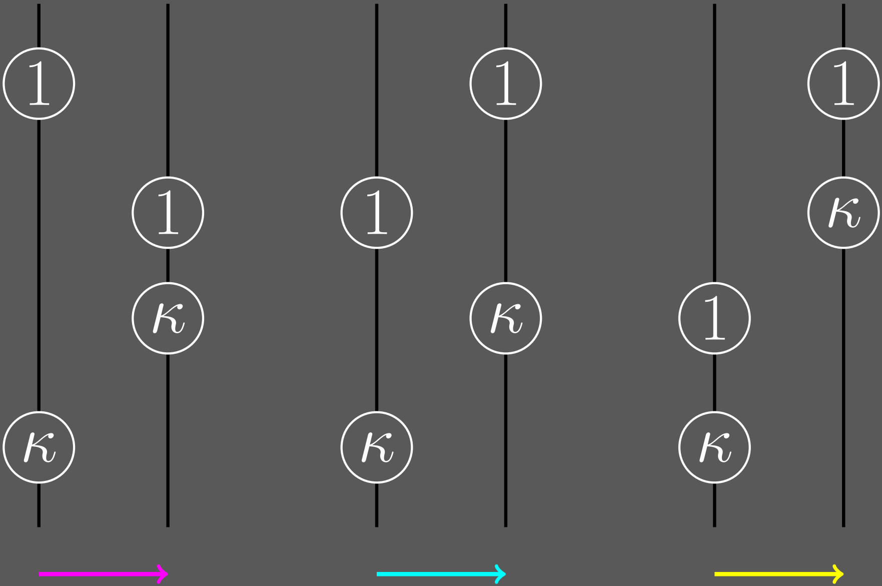
Theorem (Erdős, Hajnal, 1971)

$2^\kappa = \kappa^+$ implies that $\kappa^{+2} \not\rightarrow (\kappa^{+2}, 3)$.

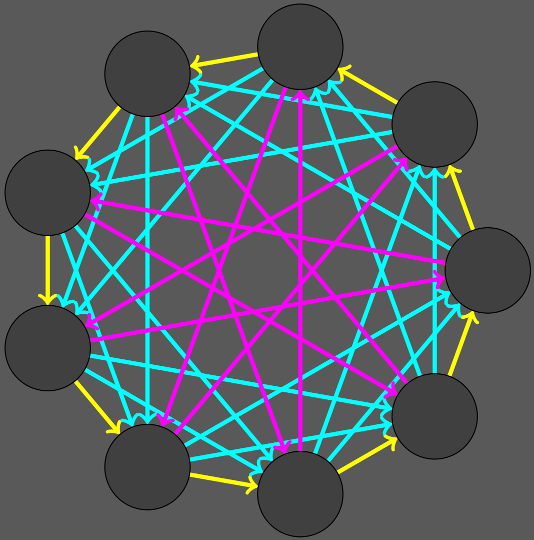
An analogue theorem

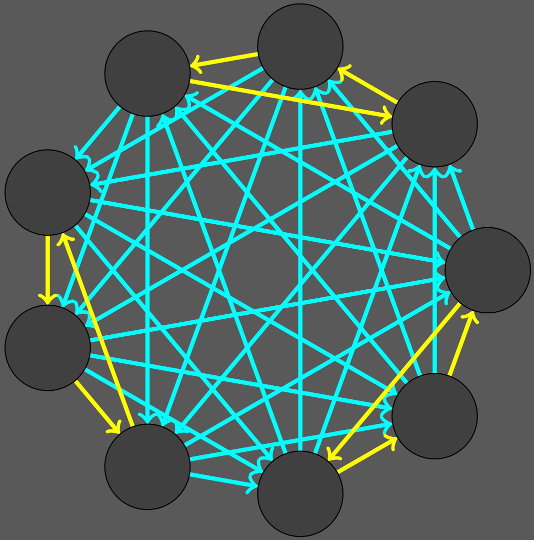


An analogue theorem



Two counterexamples





Theorem (W.)

$$\forall n \in \omega \setminus 2 : r(I_n, A_3) \leqslant \frac{(2n+1)(n^2+4n-6)}{3}.$$

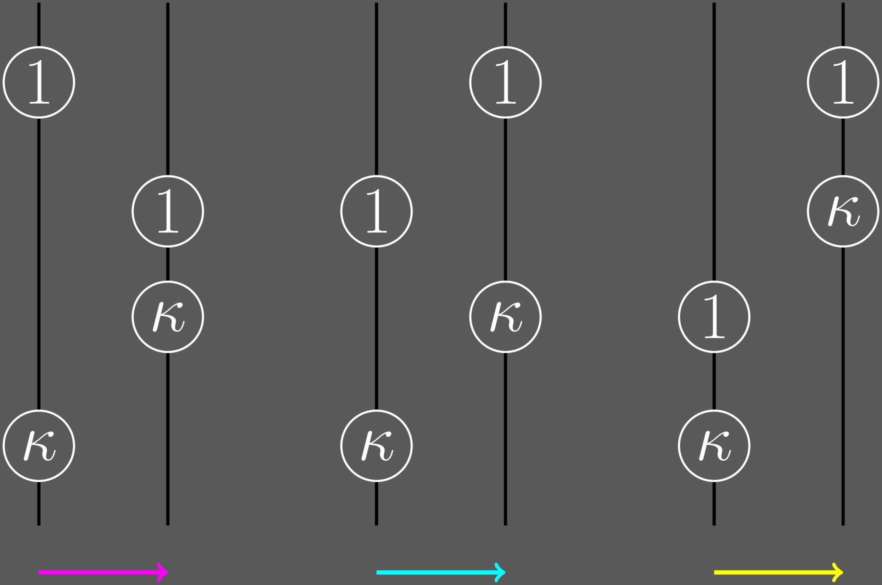
Corollary

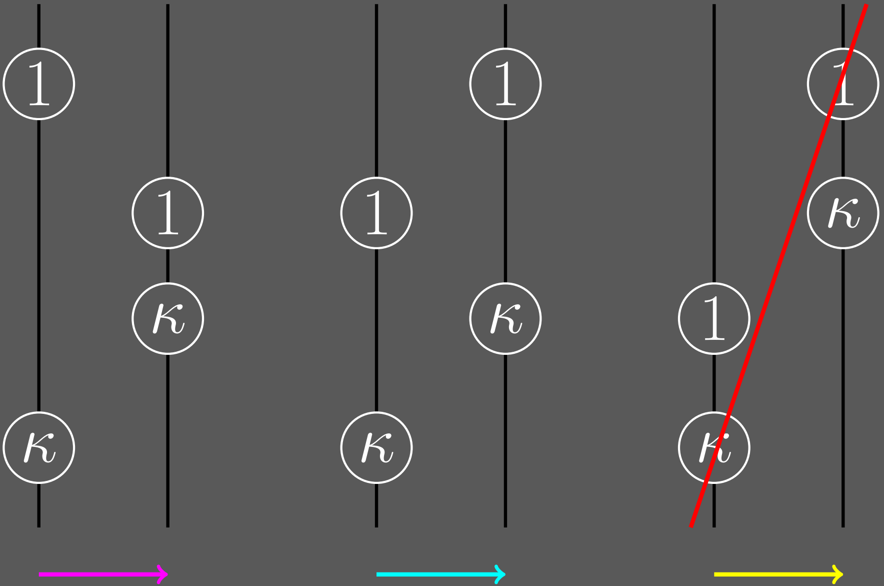
$r(I_2, A_3) = 10.$

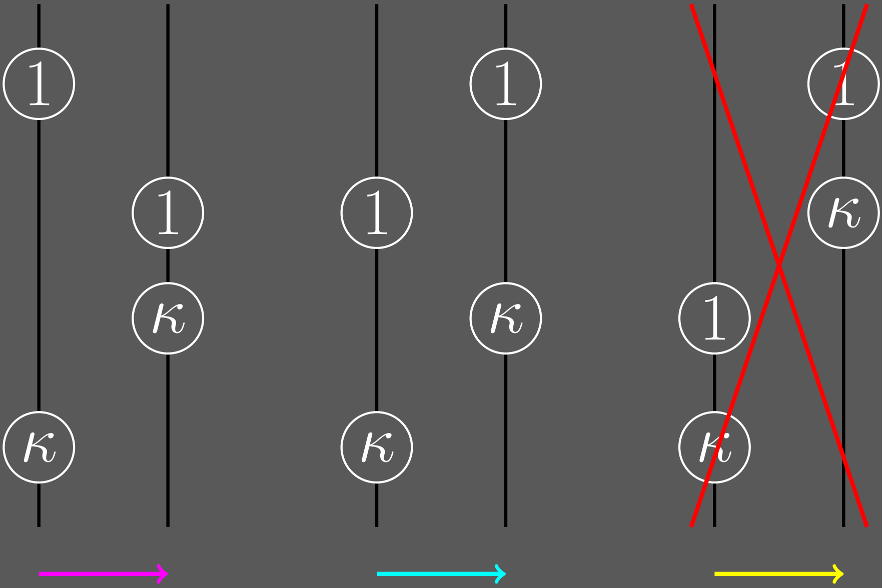
Remark

We have $r(n, 3), r(I_n, L_3) \in \mathcal{O}(n^2)$.

A variation

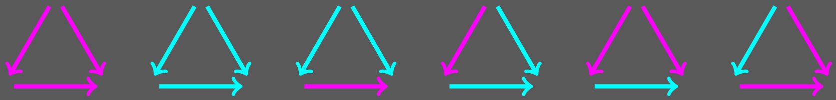



1
1
κ
κ
1
1
κ
κ
1
1
κ
κ


1
κ
1
κ
1
κ
1
κ
1
κ
1
κ

Definition

A triple is called *strongly agreeable* if and only if it is agreeable and does not contain any yellow arrow. So it is strongly agreeable precisely if it is one of these:



Notation

$r(I_l, S_m)$ is the least n such that any arc-2-coloured digraph on n vertices contains an independent set of size l or an induced subtournament of size m all induced 3-person-subtournaments of which are strongly agreeable.

Theorem (W.)

Let κ *be weakly compact, let* $\lambda \in \kappa \setminus \omega$ *be a cardinal and let* $l, m < \omega$. *Then* $r(\kappa\lambda l, m) = \kappa\lambda r(I_l, S_m)$.

Theorem (W.)

Assume $\mathrm{MA}_{\aleph_1}$. *Then for all* $l, m < \omega$ *we have* $r(\omega_1\omega l, m) = \omega_1\omega r(I_l, S_m)$.

Theorem (W.)

For all $m \in \omega \setminus 2$ *and all* $n \in \omega \setminus 3$ *we have* $r(I_m, S_n) \leqslant u(m, n)$ *where*

$$u(m,n) := \frac{1}{4}\left(3 + \sum_{i=0}^{n-1} \binom{i+m-2}{i} 4^i\right)$$

Theorem (W.)

For all $m \in \omega \setminus 2$ and all $n \in \omega \setminus 3$ we have $r(I_m, S_n) \leqslant u(m,n)$ where

$$u(m,n) := \frac{1}{4}\left(3 + \sum_{i=0}^{n-1} \binom{i+m-2}{i} 4^i\right)$$

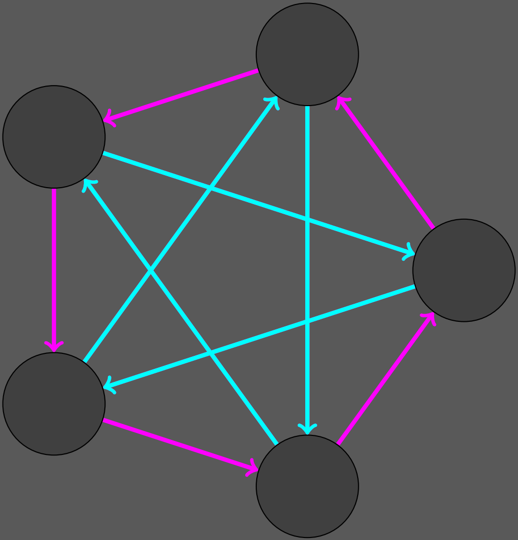
Corollary

For all $m \in \omega \setminus 3$ we have $r(I_m, S_3) \leqslant m(2m-1)$.

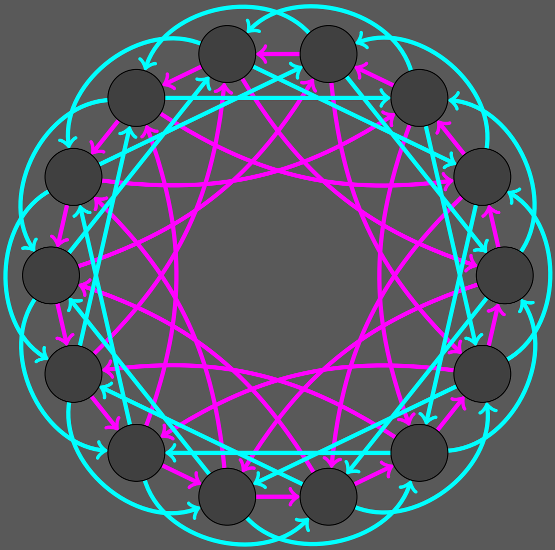
Corollary

$$\text{For any } n \in \omega \setminus 3 \text{ we have } r(I_2, S_n) \leqslant \frac{4^{n-1}+2}{3}.$$

Two other counterexamples



Two other counterexamples



	3	4	5	6	7	8	9	m
3	6	9	14	18	23	28	36	
4	9	18	25					
λ	λ	λ	λ	λ	λ	λ	λ	λ
$\lambda 2$	$\lambda 4$	$\lambda 8$	$\lambda 14$	$\lambda 28$				
$\lambda 3$	$\lambda 9$							
ω^2	ω^2	ω^2	ω^2	ω^2	ω^2	ω^2	ω^2	ω^2
$\omega^2 2$	$\omega^2 10$							
ω^3	ω^4	ω^4	ω^5	ω^5	ω^5	ω^5	ω^6	$\omega^{2+\lceil \mathrm{ld}(m) \rceil}$
ω^4	ω^7	ω^7	ω^{10}	ω^{10}	ω^{10}	ω^{10}		
ω^{5+l}	ω^{9+2l}	ω^{9+2l}	ω^{13+3l}	ω^{13+3l}	ω^{13+3l}	ω^{13+3l}	ω^{17+4l}	$\omega^{1+(4+l)\lceil \mathrm{ld}(m) \rceil}$
ω^ω	ω^ω	ω^ω	ω^ω	ω^ω	ω^ω	ω^ω	ω^ω	ω^ω
ω^{ω^2}	ω^{ω^2}	ω^{ω^2}						
$\kappa\lambda 2$	$\kappa\lambda 6$							
$\kappa\lambda 3$	$\kappa\lambda 15$							

Question

What are $\mathcal{O}(r(I_n, A_3))$ *and* $\Omega(r(I_n, A_3))$*?*

Remark

Proving lower bounds is often difficult.

Example (Jeong Han Kim, 1995)

$r(n,3) \in \Theta(\frac{n^2}{\log n})$

Example (Noga Alon, 2005)

$r(n,3,3) \in \Theta(\frac{n^3}{\operatorname{polylog} n})$.

Question

Assume that $r(\omega_1\omega, m) = \omega_1\omega$ *for all* $m < \omega$*. Does it follow that* $r(\omega_1\omega l, m) = \omega_1\omega r(I_l, S_m)$ *for all* $l, m < \omega$*?*

Thank you very much for your attention!