

# Variants of the Borel Conjecture and Sacks dense ideals

Wolfgang Wohofsky

Vienna University of Technology (TU Wien)  
and  
Kurt Gödel Research Center, Vienna (KGRC)

wolfgang.wohofsky@gmx.at

Trends in set theory  
Warsaw, Poland, July 08-11, 2012

# Outline of the talk

## ① Special sets of real numbers, Borel Conjecture

- ▶ strong measure zero, strongly meager
- ▶ Borel Conjecture, dual Borel Conjecture,  $\text{Con}(\text{BC} + \text{dBC})$

## ② Another variant of the Borel Conjecture

- ▶ Marczewski ideal  $s_0$ , “Marczewski Borel Conjecture”
- ▶ ... investigating “Sacks dense ideals”

# Outline of the talk

- ① Special sets of real numbers, Borel Conjecture
  - ▶ strong measure zero, strongly meager
  - ▶ Borel Conjecture, dual Borel Conjecture, Con(BC + dBC)
- ② Another variant of the Borel Conjecture
  - ▶ Marczewski ideal  $s_0$ , “Marczewski Borel Conjecture”
  - ▶ ... investigating “Sacks dense ideals”

# Special sets of real numbers, Borel Conjecture

## ① Special sets of real numbers, Borel Conjecture

- ▶ strong measure zero, strongly meager
- ▶ Borel Conjecture, dual Borel Conjecture,  $\text{Con}(\text{BC} + \text{dBC})$

## ② Another variant of the Borel Conjecture

- ▶ Marczewski ideal  $s_0$ , “Marczewski Borel Conjecture”
- ▶ ... investigating “Sacks dense ideals”

# The real numbers: topology, measure, algebraic structure

## The real numbers ("the reals")

- $\mathbb{R}$ , the classical real line
- $2^\omega$ , the Cantor space (totally disconnected, compact)

Structure on the reals:

- natural **topology** (intervals/basic clopen sets form a basis)
- standard (Lebesgue) **measure**
- **group structure**
  - ▶  $(2^\omega, +)$  is a topological group, with  $+$  bitwise modulo 2
- Two translation-invariant  $\sigma$ -ideals
  - ▶ meager sets  $\mathcal{M}$
  - ▶ measure zero sets  $\mathcal{N}$

# The real numbers: topology, measure, algebraic structure

## The real numbers ("the reals")

- $\mathbb{R}$ , the classical real line
- $2^\omega$ , the Cantor space (totally disconnected, compact)

Structure on the reals:

- natural **topology** (intervals/basic clopen sets form a basis)
- standard (Lebesgue) **measure**
- **group structure**
  - ▶  $(2^\omega, +)$  is a topological group, with  $+$  bitwise modulo 2
- Two translation-invariant  $\sigma$ -ideals
  - ▶ meager sets  $\mathcal{M}$
  - ▶ measure zero sets  $\mathcal{N}$

# The real numbers: topology, measure, algebraic structure

## The real numbers ("the reals")

- $\mathbb{R}$ , the classical real line
- $2^\omega$ , the Cantor space (totally disconnected, compact)

Structure on the reals:

- natural **topology** (intervals/basic clopen sets form a basis)
- standard (Lebesgue) **measure**
- **group structure**
  - ▶  $(2^\omega, +)$  is a topological group, with  $+$  bitwise modulo 2
- Two translation-invariant  $\sigma$ -ideals
  - ▶ meager sets  $\mathcal{M}$
  - ▶ measure zero sets  $\mathcal{N}$

# Strong measure zero sets

For an interval  $I \subseteq \mathbb{R}$ , let  $\lambda(I)$  denote its length.

## Definition (well-known)

A set  $X \subseteq \mathbb{R}$  is (Lebesgue) **measure zero** ( $X \in \mathcal{N}$ ) iff  
for each positive real number  $\varepsilon > 0$

there is a sequence of intervals  $(I_n)_{n < \omega}$  of total length  $\sum_{n < \omega} \lambda(I_n) \leq \varepsilon$   
such that  $X \subseteq \bigcup_{n < \omega} I_n$ .

## Definition (Borel; 1919)

A set  $X \subseteq \mathbb{R}$  is **strong measure zero** ( $X \in \mathcal{SN}$ ) iff

for each sequence of positive real numbers  $(\varepsilon_n)_{n < \omega}$

there is a sequence of intervals  $(I_n)_{n < \omega}$  with  $\forall n \in \omega \lambda(I_n) \leq \varepsilon_n$

such that  $X \subseteq \bigcup_{n < \omega} I_n$ .



# Strong measure zero sets

For an interval  $I \subseteq \mathbb{R}$ , let  $\lambda(I)$  denote its length.

## Definition (well-known)

A set  $X \subseteq \mathbb{R}$  is (Lebesgue) **measure zero** ( $X \in \mathcal{N}$ ) iff  
for each positive real number  $\varepsilon > 0$

there is a sequence of intervals  $(I_n)_{n < \omega}$  of total length  $\sum_{n < \omega} \lambda(I_n) \leq \varepsilon$   
such that  $X \subseteq \bigcup_{n < \omega} I_n$ .

## Definition (Borel; 1919)

A set  $X \subseteq \mathbb{R}$  is **strong measure zero** ( $X \in \mathcal{SN}$ ) iff  
for each sequence of positive real numbers  $(\varepsilon_n)_{n < \omega}$

there is a sequence of intervals  $(I_n)_{n < \omega}$  with  $\forall n \in \omega \lambda(I_n) \leq \varepsilon_n$   
such that  $X \subseteq \bigcup_{n < \omega} I_n$ .

# Equivalent characterization of strong measure zero sets

For  $Y, Z \subseteq 2^\omega$ , let  $Y + Z = \{y + z : y \in Y, z \in Z\}$ .

## Key Theorem (Galvin, Mycielski, Solovay; 1973)

A set  $Y \subseteq 2^\omega$  is strong measure zero if and only if for every meager set  $M \in \mathcal{M}$ ,  $Y + M \neq 2^\omega$ .

Note that  $Y + M \neq 2^\omega$  if and only if  $Y$  can be “translated away” from  $M$ , i.e., there exists a  $t \in 2^\omega$  such that  $(Y + t) \cap M = \emptyset$ .

## Key Definition

Let  $\mathcal{J} \subseteq \mathcal{P}(2^\omega)$  be arbitrary. Define

$$\mathcal{J}^* := \{Y \subseteq 2^\omega : Y + Z \neq 2^\omega \text{ for every set } Z \in \mathcal{J}\}.$$

$\mathcal{J}^*$  is the collection of “ $\mathcal{J}$ -shiftable sets”,  
i.e.,  $Y \in \mathcal{J}^*$  iff  $Y$  can be translated away from every set in  $\mathcal{J}$ .

# Equivalent characterization of strong measure zero sets

For  $Y, Z \subseteq 2^\omega$ , let  $Y + Z = \{y + z : y \in Y, z \in Z\}$ .

## Key Theorem (Galvin, Mycielski, Solovay; 1973)

A set  $Y \subseteq 2^\omega$  is strong measure zero if and only if for every meager set  $M \in \mathcal{M}$ ,  $Y + M \neq 2^\omega$ .

Note that  $Y + M \neq 2^\omega$  if and only if  $Y$  can be “translated away” from  $M$ , i.e., there exists a  $t \in 2^\omega$  such that  $(Y + t) \cap M = \emptyset$ .

## Key Definition

Let  $\mathcal{J} \subseteq \mathcal{P}(2^\omega)$  be arbitrary. Define

$$\mathcal{J}^* := \{Y \subseteq 2^\omega : Y + Z \neq 2^\omega \text{ for every set } Z \in \mathcal{J}\}.$$

$\mathcal{J}^*$  is the collection of “ $\mathcal{J}$ -shiftable sets”,  
i.e.,  $Y \in \mathcal{J}^*$  iff  $Y$  can be translated away from every set in  $\mathcal{J}$ .

# Strongly meager sets

## Key Definition (from previous slide)

Let  $\mathcal{J} \subseteq \mathcal{P}(2^\omega)$  be arbitrary. Define

$$\mathcal{J}^* := \{Y \subseteq 2^\omega : Y + Z \neq 2^\omega \text{ for every set } Z \in \mathcal{J}\}.$$

## Key Theorem (Galvin, Mycielski, Solovay; 1973)

A set  $Y$  is **strong measure zero** if and only if it is “ $\mathcal{M}$ -shiftable”, i.e.,

$$SN = \mathcal{M}^*$$

Replacing  $\mathcal{M}$  by  $\mathcal{N}$  yields a notion *dual to strong measure zero*:

## Definition

A set  $Y$  is **strongly meager** ( $Y \in SM$ ) iff it is “ $\mathcal{N}$ -shiftable”, i.e.,

$$SM := \mathcal{N}^*$$

# Strongly meager sets

## Key Definition (from previous slide)

Let  $\mathcal{J} \subseteq \mathcal{P}(2^\omega)$  be arbitrary. Define

$$\mathcal{J}^* := \{Y \subseteq 2^\omega : Y + Z \neq 2^\omega \text{ for every set } Z \in \mathcal{J}\}.$$

## Key Theorem (Galvin, Mycielski, Solovay; 1973)

A set  $Y$  is **strong measure zero** if and only if it is “ **$\mathcal{M}$ -shiftable**”, i.e.,

$$SN = \mathcal{M}^*$$

Replacing  $\mathcal{M}$  by  $\mathcal{N}$  yields a notion *dual to strong measure zero*:

## Definition

A set  $Y$  is **strongly meager** ( $Y \in SM$ ) iff it is “ **$\mathcal{N}$ -shiftable**”, i.e.,

$$SM := \mathcal{N}^*$$

# Strongly meager sets

## Key Definition (from previous slide)

Let  $\mathcal{J} \subseteq \mathcal{P}(2^\omega)$  be arbitrary. Define

$$\mathcal{J}^* := \{Y \subseteq 2^\omega : Y + Z \neq 2^\omega \text{ for every set } Z \in \mathcal{J}\}.$$

## Key Theorem (Galvin, Mycielski, Solovay; 1973)

A set  $Y$  is **strong measure zero** if and only if it is “ **$\mathcal{M}$ -shiftable**”, i.e.,

$$SN = \mathcal{M}^*$$

Replacing  $\mathcal{M}$  by  $\mathcal{N}$  yields a notion *dual to strong measure zero*:

## Definition

A set  $Y$  is **strongly meager** ( $Y \in SM$ ) iff it is “ **$\mathcal{N}$ -shiftable**”, i.e.,

$$SM := \mathcal{N}^*$$

# Borel Conjecture + dual Borel Conjecture

## Definition

The **Borel Conjecture** (BC) is the statement that there are **no** uncountable strong measure zero sets, i.e.,  $\mathcal{SN} = \mathcal{M}^* = [2^\omega]^{\leq \aleph_0}$ .

- **Con(BC)**, actually BC holds in the Laver model (Laver, 1976)

## Definition

The **dual Borel Conjecture** (dBC) is the statement that there are **no** uncountable strongly meager sets, i.e.,  $\mathcal{SM} = \mathcal{N}^* = [2^\omega]^{\leq \aleph_0}$ .

- **Con(dBC)**, actually dBC holds in the Cohen model (Carlson, 1993)

## Theorem (Goldstern, Kellner, Shelah, W.; 2011)

There is a model of ZFC in which both the Borel Conjecture and the dual Borel Conjecture hold, i.e., **Con(BC + dBC)**.

# Borel Conjecture + dual Borel Conjecture

## Definition

The **Borel Conjecture** (BC) is the statement that there are **no** uncountable strong measure zero sets, i.e.,  $\mathcal{SN} = \mathcal{M}^* = [2^\omega]^{\leq \aleph_0}$ .

- **Con(BC)**, actually BC holds in the Laver model (Laver, 1976)

## Definition

The **dual Borel Conjecture** (dBC) is the statement that there are **no** uncountable strongly meager sets, i.e.,  $\mathcal{SM} = \mathcal{N}^* = [2^\omega]^{\leq \aleph_0}$ .

- **Con(dBC)**, actually dBC holds in the Cohen model (Carlson, 1993)

Theorem (Goldstern, Kellner, Shelah, W.; 2011)

There is a model of ZFC in which both the Borel Conjecture and the dual Borel Conjecture hold, i.e., **Con(BC + dBC)**.



# Borel Conjecture + dual Borel Conjecture

## Definition

The **Borel Conjecture** (BC) is the statement that there are **no** uncountable strong measure zero sets, i.e.,  $\mathcal{SN} = \mathcal{M}^* = [2^\omega]^{\leq \aleph_0}$ .

- **Con(BC)**, actually BC holds in the Laver model (Laver, 1976)

## Definition

The **dual Borel Conjecture** (dBC) is the statement that there are **no** uncountable strongly meager sets, i.e.,  $\mathcal{SM} = \mathcal{N}^* = [2^\omega]^{\leq \aleph_0}$ .

- **Con(dBC)**, actually dBC holds in the Cohen model (Carlson, 1993)

## Theorem (Goldstern, Kellner, Shelah, W.; 2011)

There is a model of ZFC in which both the Borel Conjecture and the dual Borel Conjecture hold, i.e., **Con(BC + dBC)**.



# Small subsets of the real line and generalizations of the Borel Conjecture

Wolfgang Wohofsky (advisor: Martin Goldstern)

Recipient of a DOC-fellowship of the Austrian Academy of Sciences at the Institute of Discrete Mathematics and Geometry

26.02.2010

## Cohen and the Continuum Hypothesis



Paul J. Cohen (1934-2007)

Hebrew origin of the continuum

$$2^{\aleph_0} = \aleph_1$$

$$2^{\aleph_1} = \aleph_2$$

$$2^{\aleph_2} = \aleph_3$$

In 1937, Georg Cantor showed that the set of real numbers is uncountable. It is not a solution, but surely better than the set of natural numbers (counting numbers). There is an interesting correspondence between the set of real numbers and the hypothesis of set theory.

A couple of years later Cohen proposed the so-called **continuum hypothesis** (CH), which asserts that the set of real numbers "the continuum" has the best possible size.

For some years it failed to gain a lot of interest. David Hilbert listed it as the most important and interesting unsolved problem of the 20th century. Cohen's CH is now on top of the list of the 23 open problems which are presented at the Paris colloquium of the International Congress of Mathematicians in 2002.

Then the problem turned. In 1963, the Israeli-born American logician Paul Cohen showed that the continuum hypothesis cannot be proved within the assumptions ZFC. Paul Cohen, who died on March 29, 2006, is credited the technique of **forcing**. In 1965, he also showed that continuum power CH from ZFC, thereby resolving the independence of the continuum hypothesis.

### The New York Times

Nov 14th, 1963

#### PAUL COHEN HAS PROVED THAT THE CONTINUUM HYPOTHESIS IS UNDECIDABLE

Paul Cohen has proved that the continuum hypothesis is undecidable. This means that it cannot be proved or disproved within the framework of the current mathematical axioms.

... (transcription of the article) ...



Georg Cantor (1845-1918)



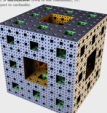
David Hilbert (1862-1943)



Kurt Gödel (1905-1982)

## Small sets of real numbers

Menger spaces (also called  $\mathfrak{M}$ -spaces) are a class of topological spaces. They are defined as follows: A topological space  $X$  is called a Menger space if for every countable family  $\{U_n\}_{n \in \mathbb{N}}$  of open sets, there exists a countable subfamily  $\{V_n\}_{n \in \mathbb{N}}$  such that  $\bigcup_{n \in \mathbb{N}} V_n = X$ .



When is a set  $X$  of real numbers countable? It is not a trivial question. It depends on the topological structure. The fundamental axiom of set theory, ZFC, does not allow us to answer this question. There are sets which are neither countable nor uncountable.

- A set  $X$  is meager if it is the union of only countably many nowhere dense sets.
- A set  $X$  is meager if it is the union of only countably many nowhere dense sets.
- A set  $X$  is meager if it is the union of only countably many nowhere dense sets.

$$\sum_{n \in \mathbb{N}} \text{meag}(X_n) = \text{meag}\left(\bigcup_{n \in \mathbb{N}} X_n\right)$$

Locally splitting the measure of a set  $X$  in  $\mathbb{R}^n$  by its length (or area or volume) is not possible. This is the content of the Borel Conjecture. It is a deep result in set theory. It is a deep result in set theory. It is a deep result in set theory.

## Even smaller sets and the (dual) Borel Conjecture

Can we strengthen the notion of being measure zero (and stronger repressibility)?

Yes! If it is strongly measure zero (SMZ), then it is also measure zero. It is a stronger notion. It is a stronger notion. It is a stronger notion.

**Dual Conjecture**: The only SMZ set is the countable set ( $\aleph_0$ ).

**And Dual Conjecture**: The only SMZ set is the countable set ( $\aleph_0$ ).

## Shelah's oracle c.c.c. forcing



Saharon Shelah (1945-2021)

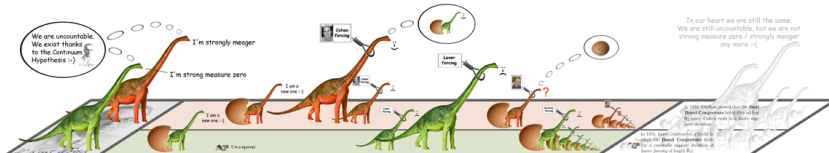
Shelah's forcing is a type of forcing. It is a type of forcing. It is a type of forcing. It is a type of forcing.

Shelah's forcing is a type of forcing. It is a type of forcing. It is a type of forcing. It is a type of forcing.

Shelah's forcing is a type of forcing. It is a type of forcing. It is a type of forcing. It is a type of forcing.

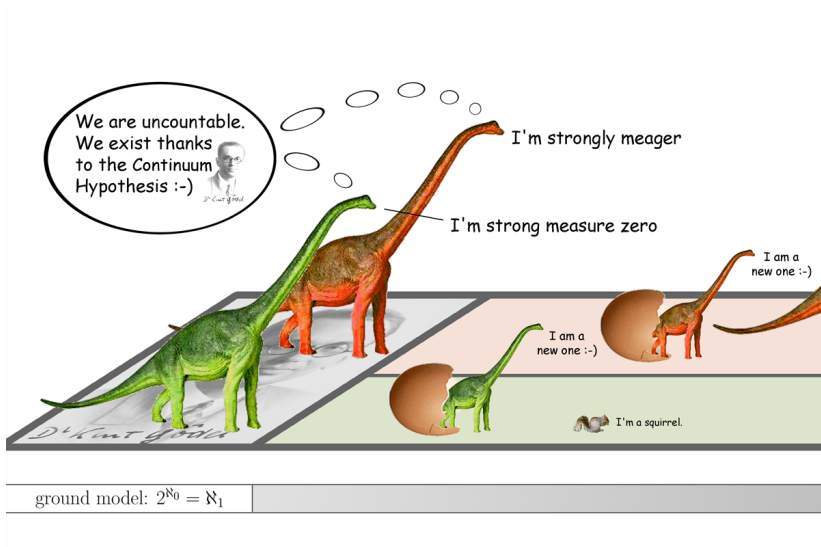
Shelah's forcing is a type of forcing. It is a type of forcing. It is a type of forcing. It is a type of forcing.

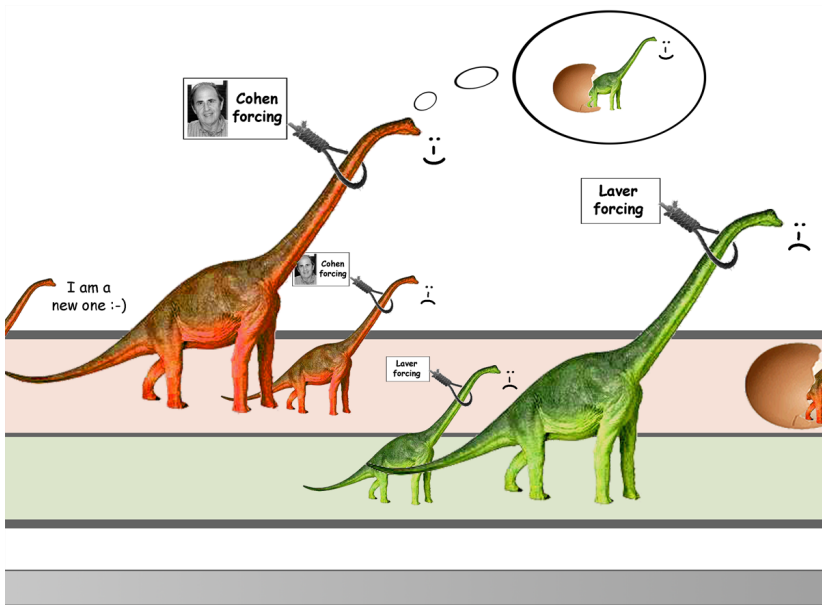
Shelah's forcing is a type of forcing. It is a type of forcing. It is a type of forcing. It is a type of forcing.

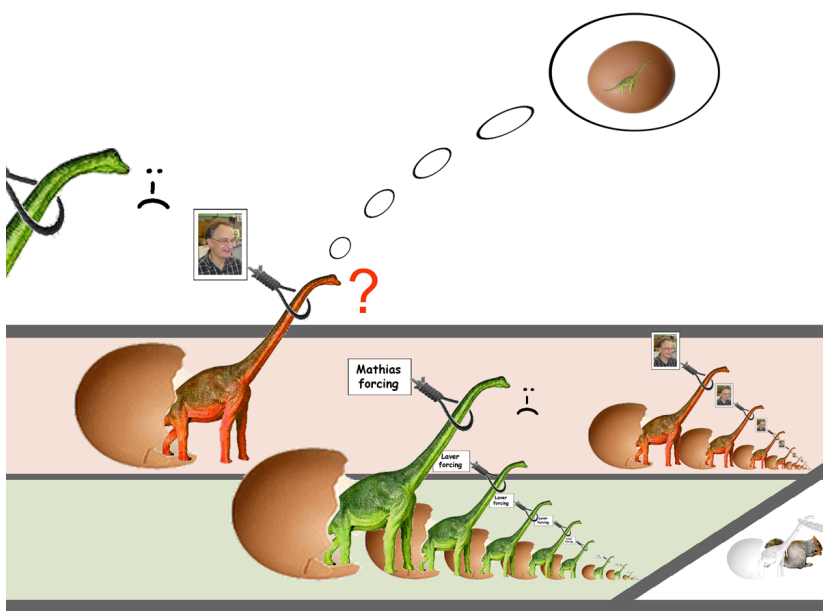


ground model:  $2^{\aleph_0} = \aleph_1$

final model:  $2^{\aleph_0} = \aleph_2$







# Another variant of the Borel Conjecture

- ① Special sets of real numbers, Borel Conjecture
  - ▶ strong measure zero, strongly meager
  - ▶ Borel Conjecture, dual Borel Conjecture, Con(BC + dBC)
- ② **Another variant of the Borel Conjecture**
  - ▶ Marczewski ideal  $s_0$ , “Marczewski Borel Conjecture”
  - ▶ ... investigating “Sacks dense ideals”

# Marczewski Borel Conjecture (MBC)

Assume that  $\mathcal{J} \subseteq \mathcal{P}(2^\omega)$  is a translation-invariant  $\sigma$ -ideal. Recall that

$$\mathcal{J}^* := \{Y \subseteq 2^\omega : Y + Z \neq 2^\omega \text{ for every set } Z \in \mathcal{J}\}.$$

## Definition

The  **$\mathcal{J}$ -Borel Conjecture** ( $\mathcal{J}$ -BC) is the statement that there are **no** uncountable  $\mathcal{J}$ -shiftable sets, i.e.,  $\mathcal{J}^* = [2^\omega]^{\leq \omega}$ .

The **Marczewski ideal**  $s_0$  is the collection of all  $Z \subseteq 2^\omega$  such that for each perfect set  $P$ , there exists a perfect subset  $Q \subseteq P$  with  $Q \cap Z = \emptyset$ .

## Definition

The **Marczewski Borel Conjecture** (MBC) is the statement that there are **no** uncountable  $s_0$ -shiftable sets, i.e.,  $s_0^* = [2^\omega]^{\leq \omega}$ .

What about  $\text{Con}(\text{MBC})$ ?

Can MBC be forced?

# Marczewski Borel Conjecture (MBC)

Assume that  $\mathcal{J} \subseteq \mathcal{P}(2^\omega)$  is a translation-invariant  $\sigma$ -ideal. Recall that

$$\mathcal{J}^* := \{Y \subseteq 2^\omega : Y + Z \neq 2^\omega \text{ for every set } Z \in \mathcal{J}\}.$$

## Definition

The  **$\mathcal{J}$ -Borel Conjecture** ( $\mathcal{J}$ -BC) the statement that there are **no** uncountable  $\mathcal{J}$ -shiftable sets, i.e.,  $\mathcal{J}^* = [2^\omega]^{\leq \omega}$ .

The **Marczewski ideal**  $s_0$  is the collection of all  $Z \subseteq 2^\omega$  such that for each perfect set  $P$ , there exists a perfect subset  $Q \subseteq P$  with  $Q \cap Z = \emptyset$ .

## Definition

The **Marczewski Borel Conjecture** (MBC) is the statement that there are **no** uncountable  $s_0$ -shiftable sets, i.e.,  $s_0^* = [2^\omega]^{\leq \omega}$ .

What about  $\text{Con}(\text{MBC})$ ?

Can MBC be forced?



# Marczewski Borel Conjecture (MBC)

Assume that  $\mathcal{J} \subseteq \mathcal{P}(2^\omega)$  is a translation-invariant  $\sigma$ -ideal. Recall that

$$\mathcal{J}^* := \{Y \subseteq 2^\omega : Y + Z \neq 2^\omega \text{ for every set } Z \in \mathcal{J}\}.$$

## Definition

The  **$\mathcal{J}$ -Borel Conjecture** ( $\mathcal{J}$ -BC) the statement that there are **no** uncountable  $\mathcal{J}$ -shiftable sets, i.e.,  $\mathcal{J}^* = [2^\omega]^{\leq \omega}$ .

The **Marczewski ideal**  $s_0$  is the collection of all  $Z \subseteq 2^\omega$  such that for each perfect set  $P$ , there exists a perfect subset  $Q \subseteq P$  with  $Q \cap Z = \emptyset$ .

## Definition

The **Marczewski Borel Conjecture** (MBC) is the statement that there are **no** uncountable  $s_0$ -shiftable sets, i.e.,  $s_0^* = [2^\omega]^{\leq \omega}$ .

What about  $\text{Con}(\text{MBC})$ ?

Can MBC be forced?

# Marczewski Borel Conjecture (MBC)

Assume that  $\mathcal{J} \subseteq \mathcal{P}(2^\omega)$  is a translation-invariant  $\sigma$ -ideal. Recall that

$$\mathcal{J}^* := \{Y \subseteq 2^\omega : Y + Z \neq 2^\omega \text{ for every set } Z \in \mathcal{J}\}.$$

## Definition

The  **$\mathcal{J}$ -Borel Conjecture** ( $\mathcal{J}$ -BC) the statement that there are **no** uncountable  $\mathcal{J}$ -shiftable sets, i.e.,  $\mathcal{J}^* = [2^\omega]^{\leq \omega}$ .

The **Marczewski ideal**  $s_0$  is the collection of all  $Z \subseteq 2^\omega$  such that for each perfect set  $P$ , there exists a perfect subset  $Q \subseteq P$  with  $Q \cap Z = \emptyset$ .

## Definition

The **Marczewski Borel Conjecture** (MBC) is the statement that there are **no** uncountable  $s_0$ -shiftable sets, i.e.,  $s_0^* = [2^\omega]^{\leq \omega}$ .

What about  $\text{Con}(\text{MBC})$ ?

Can MBC be forced?

# Marczewski Borel Conjecture (MBC)

Assume that  $\mathcal{J} \subseteq \mathcal{P}(2^\omega)$  is a translation-invariant  $\sigma$ -ideal. Recall that

$$\mathcal{J}^* := \{Y \subseteq 2^\omega : Y + Z \neq 2^\omega \text{ for every set } Z \in \mathcal{J}\}.$$

## Definition

The  **$\mathcal{J}$ -Borel Conjecture** ( $\mathcal{J}$ -BC) is the statement that there are **no** uncountable  $\mathcal{J}$ -shiftable sets, i.e.,  $\mathcal{J}^* = [2^\omega]^{\leq \omega}$ .

The **Marczewski ideal**  $s_0$  is the collection of all  $Z \subseteq 2^\omega$  such that for each perfect set  $P$ , there exists a perfect subset  $Q \subseteq P$  with  $Q \cap Z = \emptyset$ .

## Definition

The **Marczewski Borel Conjecture** (MBC) is the statement that there are **no** uncountable  $s_0$ -shiftable sets, i.e.,  $s_0^* = [2^\omega]^{\leq \omega}$ .

What about  $\text{Con}(\text{MBC})$ ?

Can MBC be forced?



# Sacks dense ideals

Unlike BC and dBC, the status of MBC under CH is unclear. . .

- Is MBC (i.e.,  $s_0^* = [2^\omega]^{\leq \aleph_0}$ ) consistent with CH?
- Or does CH even imply MBC?

To investigate the situation under CH, I introduced the following notion:

## Definition

A collection  $\mathcal{I} \subseteq \mathcal{P}(2^\omega)$  is a **Sacks dense ideal** (S.d.i.) iff

- $\mathcal{I}$  is a  $\sigma$ -ideal
- $\mathcal{I}$  is *translation-invariant*
- $\mathcal{I}$  is **dense in Sacks forcing**, more explicitly, for each perfect  $P \subseteq 2^\omega$ , there is a perfect subset  $Q$  in the ideal, i.e.,  $\exists Q \subseteq P, Q \in \mathcal{I}$

## Lemma (“Main Lemma”)

Assume CH. Let  $\mathcal{I}$  be a Sacks dense ideal. Then  $s_0^* \subseteq \mathcal{I}$ .

# Sacks dense ideals

Unlike BC and dBC, the status of MBC under CH is unclear. . .

- Is MBC (i.e.,  $s_0^* = [2^\omega]^{\leq \aleph_0}$ ) consistent with CH?
- Or does CH even imply MBC?

To investigate the situation under CH, I introduced the following notion:

## Definition

A collection  $\mathcal{I} \subseteq \mathcal{P}(2^\omega)$  is a **Sacks dense ideal** (S.d.i.) iff

- $\mathcal{I}$  is a  $\sigma$ -ideal
- $\mathcal{I}$  is *translation-invariant*
- $\mathcal{I}$  is **dense in Sacks forcing**, more explicitly, for each perfect  $P \subseteq 2^\omega$ , there is a perfect subset  $Q$  in the ideal, i.e.,  $\exists Q \subseteq P, Q \in \mathcal{I}$

## Lemma (“Main Lemma”)

Assume CH. Let  $\mathcal{I}$  be a Sacks dense ideal. Then  $s_0^* \subseteq \mathcal{I}$ .

# Sacks dense ideals

Unlike BC and dBC, the status of MBC under CH is unclear. . .

- Is MBC (i.e.,  $s_0^* = [2^\omega]^{\leq \aleph_0}$ ) consistent with CH?
- Or does CH even imply MBC?

To investigate the situation under CH, I introduced the following notion:

## Definition

A collection  $\mathcal{I} \subseteq \mathcal{P}(2^\omega)$  is a **Sacks dense ideal** (S.d.i.) iff

- $\mathcal{I}$  is a  $\sigma$ -ideal
- $\mathcal{I}$  is *translation-invariant*
- $\mathcal{I}$  is **dense in Sacks forcing**, more explicitly, for each perfect  $P \subseteq 2^\omega$ , there is a perfect subset  $Q$  in the ideal, i.e.,  $\exists Q \subseteq P, Q \in \mathcal{I}$

## Lemma (“Main Lemma”)

Assume CH. Let  $\mathcal{I}$  be a Sacks dense ideal. Then  $s_0^* \subseteq \mathcal{I}$ .

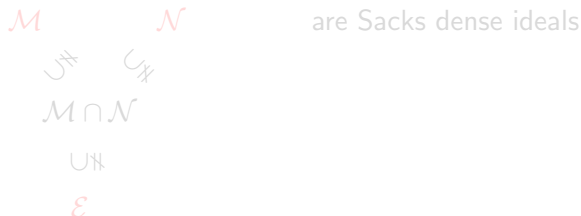
# More and more Sacks dense ideals

Lemma (“Main Lemma”; from previous slide)

Assume CH. Let  $\mathcal{I}$  be a Sacks dense ideal. Then  $s_0^* \subseteq \mathcal{I}$ .

In other words:  $s_0^* \subseteq \bigcap \{ \mathcal{I} : \mathcal{I} \text{ is a Sacks dense ideal} \}$ .

Can we (consistently) find many Sacks dense ideals under CH?



$\mathcal{SM}$  is NOT a Sacks dense ideal, BUT...



# More and more Sacks dense ideals

Lemma (“Main Lemma”; from previous slide)

Assume CH. Let  $\mathcal{I}$  be a Sacks dense ideal. Then  $s_0^* \subseteq \mathcal{I}$ .

In other words:  $s_0^* \subseteq \bigcap \{ \mathcal{I} : \mathcal{I} \text{ is a Sacks dense ideal} \}$ .

Can we (consistently) find **many Sacks dense ideals** under CH?

$\mathcal{M}$        $\mathcal{N}$       are Sacks dense ideals

$\mathcal{M} \cap \mathcal{N}$

$\mathcal{M} \cup \mathcal{N}$

$\mathcal{E}$

$\mathcal{M}$  is NOT a Sacks dense ideal, BUT...

# More and more Sacks dense ideals

Lemma (“Main Lemma”; from previous slide)

Assume CH. Let  $\mathcal{I}$  be a Sacks dense ideal. Then  $s_0^* \subseteq \mathcal{I}$ .

In other words:  $s_0^* \subseteq \bigcap \{ \mathcal{I} : \mathcal{I} \text{ is a Sacks dense ideal} \}$ .

Can we (consistently) find **many Sacks dense ideals** under CH?

$\mathcal{M}$        $\mathcal{N}$       are Sacks dense ideals

$\mathcal{M} \cup \mathcal{N}$

$\mathcal{M} \cap \mathcal{N}$

$\mathcal{M} \setminus \mathcal{N}$

$\mathcal{N} \setminus \mathcal{M}$

$\mathcal{M}$  is NOT a Sacks dense ideal, BUT...

# More and more Sacks dense ideals

Lemma (“Main Lemma”; from previous slide)

Assume CH. Let  $\mathcal{I}$  be a Sacks dense ideal. Then  $s_0^* \subseteq \mathcal{I}$ .

In other words:  $s_0^* \subseteq \bigcap \{ \mathcal{I} : \mathcal{I} \text{ is a Sacks dense ideal} \}$ .

Can we (consistently) find **many Sacks dense ideals** under CH?

$\mathcal{M}$                    $\mathcal{N}$                   are Sacks dense ideals

$\cup \#$        $\cup \#$

$\mathcal{M} \cap \mathcal{N}$

$\cup \#$

$\mathcal{E}$

$\mathcal{S}\mathcal{M}$  is NOT a Sacks dense ideal, BUT...

# More and more Sacks dense ideals

Lemma (“Main Lemma”; from previous slide)

Assume CH. Let  $\mathcal{I}$  be a Sacks dense ideal. Then  $s_0^* \subseteq \mathcal{I}$ .

In other words:  $s_0^* \subseteq \bigcap \{ \mathcal{I} : \mathcal{I} \text{ is a Sacks dense ideal} \}$ .

Can we (consistently) find **many Sacks dense ideals** under CH?

$\mathcal{M}$                    $\mathcal{N}$                   are Sacks dense ideals

$\cup \#$                    $\cup \#$

$\mathcal{M} \cap \mathcal{N}$

$\cup \#$

$\mathcal{E}$

$\mathcal{S}\mathcal{M}$  is NOT a Sacks dense ideal, BUT...

$\mathcal{E}$  $\cup \aleph_1$  $\bigcap \{\mathcal{I}_f : f \in \omega^\omega\} \subseteq \text{null-additive} \subseteq \mathcal{SN} \cap \mathcal{SM}$  $\cup \aleph_1$  $\bigcap \{\mathcal{I}_f : f \in \omega^\omega\} \cap \mathcal{E}_0$  $\cup \aleph_1$  $\exists$  uncount.  $\mathcal{Y} \in \bigcap \{\mathcal{I}_\alpha : \alpha \in \omega_1\}$ , for any  $\aleph_1$ -sized family of  $\mathcal{I}_\alpha$ 's $\cup \aleph_1 \leftarrow$  proved(?) 5 days ago (using  $s_0^{\text{trans}}$ ) $\bigcap \{\mathcal{I} : \mathcal{I} \text{ is a Sacks dense ideal}\}$  $\cup \aleph_1 \leftarrow$  "Main Lemma" $s_0^*$  $\cup \aleph_1$  $[2^\omega]^{\leq \aleph_0}$

$\mathcal{E}$  $\cup \aleph_1$  $\bigcap \{\mathcal{I}_f : f \in \omega^\omega\} \subseteq \text{null-additive} \subseteq \mathcal{SN} \cap \mathcal{SM}$  $\cup \aleph_1$  $\bigcap \{\mathcal{I}_f : f \in \omega^\omega\} \cap \mathcal{E}_0$  $\cup \aleph_1$  $\exists$  uncount.  $\mathcal{Y} \in \bigcap \{\mathcal{I}_\alpha : \alpha \in \omega_1\}$ , for any  $\aleph_1$ -sized family of  $\mathcal{I}_\alpha$ 's $\cup \aleph_1 \leftarrow$  proved(?) 5 days ago (using  $s_0^{\text{trans}}$ ) $\bigcap \{\mathcal{I} : \mathcal{I} \text{ is a Sacks dense ideal}\}$  $\cup \aleph_1 \leftarrow$  "Main Lemma" $s_0^*$  $\cup \aleph_1$  $[2^\omega]^{\leq \aleph_0}$

$\mathcal{E}$  $\cup \aleph_1$  $\bigcap \{\mathcal{I}_f : f \in \omega^\omega\} \subseteq \text{null-additive} \subseteq \mathcal{SN} \cap \mathcal{SM}$  $\cup \aleph_1$  $\bigcap \{\mathcal{I}_f : f \in \omega^\omega\} \cap \mathcal{E}_0$  $\cup \aleph_1$  $\exists$  uncount.  $\mathcal{Y} \in \bigcap \{\mathcal{I}_\alpha : \alpha \in \omega_1\}$ , for any  $\aleph_1$ -sized family of  $\mathcal{I}_\alpha$ 's $\cup \aleph_1 \leftarrow$  proved(?) 5 days ago (using  $s_0^{\text{trans}}$ ) $\bigcap \{\mathcal{I} : \mathcal{I} \text{ is a Sacks dense ideal}\}$  $\cup \aleph_1 \leftarrow$  "Main Lemma" $s_0^*$  $\cup \aleph_1$  $[2^\omega]^{\leq \aleph_0}$

$\mathcal{E}$  $\cup \aleph_1$  $\bigcap \{\mathcal{I}_f : f \in \omega^\omega\} \subseteq \text{null-additive} \subseteq \mathcal{SN} \cap \mathcal{SM}$  $\cup \aleph_1$  $\bigcap \{\mathcal{I}_f : f \in \omega^\omega\} \cap \mathcal{E}_0$  $\cup \aleph_1$ 

$\exists$  uncount.  $\mathcal{Y} \in \bigcap \{\mathcal{I}_\alpha : \alpha \in \omega_1\}$ , for any  $\aleph_1$ -sized family of  $\mathcal{I}_\alpha$ 's

$\cup \aleph_1 \leftarrow$  proved(?) 5 days ago (using  $s_0^{\text{trans}}$ )

$\bigcap \{\mathcal{I} : \mathcal{I} \text{ is a Sacks dense ideal}\}$

$\cup \aleph_1 \leftarrow$  "Main Lemma"

 $s_0^*$  $\cup \aleph_1$  $[2^\omega]^{\leq \aleph_0}$



$\mathcal{E}$  $\cup \aleph_1$  $\bigcap \{\mathcal{I}_f : f \in \omega^\omega\} \subseteq \text{null-additive} \subseteq \mathcal{SN} \cap \mathcal{SM}$  $\cup \aleph_1$  $\bigcap \{\mathcal{I}_f : f \in \omega^\omega\} \cap \mathcal{E}_0$  $\cup \aleph_1$ 

$\exists$  uncount.  $\mathcal{Y} \in \bigcap \{\mathcal{I}_\alpha : \alpha \in \omega_1\}$ , for any  $\aleph_1$ -sized family of  $\mathcal{I}_\alpha$ 's

$\cup \aleph_1 \leftarrow$  proved(?) 5 days ago (using  $s_0^{\text{trans}}$ )

 $\bigcap \{\mathcal{I} : \mathcal{I} \text{ is a Sacks dense ideal}\}$ 

$\cup \aleph_1 \leftarrow$  "Main Lemma"

 $s_0^*$  $\cup \aleph_1$  $[2^\omega]^{\leq \aleph_0}$

$\mathcal{E}$  $\cup \aleph_1$  $\bigcap \{ \mathcal{I}_f : f \in \omega^\omega \} \subseteq \text{null-additive} \subseteq \mathcal{SN} \cap \mathcal{SM}$  $\cup \aleph_1$  $\bigcap \{ \mathcal{I}_f : f \in \omega^\omega \} \cap \mathcal{E}_0$  $\cup \aleph_1$ 

$\exists$  uncount.  $\mathcal{Y} \in \bigcap \{ \mathcal{I}_\alpha : \alpha \in \omega_1 \}$ , for any  $\aleph_1$ -sized family of  $\mathcal{I}_\alpha$ 's

$\cup \aleph_1 \leftarrow$  proved(?) 5 days ago (using  $s_0^{\text{trans}}$ )

 $\bigcap \{ \mathcal{I} : \mathcal{I} \text{ is a Sacks dense ideal} \}$ 

$\cup \aleph_1 \leftarrow$  "Main Lemma"

 $s_0^*$  $\cup \aleph_1$  $[2^\omega]^{\leq \aleph_0}$

## Question

Does  $[2^\omega]^{\leq \aleph_0} = \bigcap \{ \mathcal{I} : \mathcal{I} \text{ is S.d.i.} \}$  (at least consistently) hold under CH?

If yes, MBC (i.e.,  $s_0^* = [2^\omega]^{\leq \aleph_0}$ ) follows from CH (Con(MBC+CH), resp.).

## Theorem

Let  $\{ \mathcal{I}_\alpha : \alpha < \omega_1 \}$  be an  $\aleph_1$ -sized family of Sacks dense ideals. Then there exists an uncountable set  $Y \in \bigcap_{\alpha < \omega_1} \mathcal{I}_\alpha$ .

Moreover, we can construct the set  $Y$  in such a way that  $Y \notin \mathcal{J}$  for some other Sacks dense ideal  $\mathcal{J}$  (proved(?) 5 days ago (using  $s_0^{\text{trans}}$ )).

$$Y \in s_0 \quad :\iff \quad \forall p \exists q \leq p \quad |[q] \cap Y| \leq \aleph_0$$

## Definition

$$Y \in s_0^{\text{trans}} \quad :\iff \quad \forall p \exists q \leq p \quad \forall t \in 2^\omega \quad |(t + [q]) \cap Y| \leq \aleph_0$$

## Question

What can we say about the family  $s_0^{\text{trans}}$ ? Any relation to null-additive?

## Question

Does  $[2^\omega]^{\leq \aleph_0} = \bigcap \{ \mathcal{I} : \mathcal{I} \text{ is S.d.i.} \}$  (at least consistently) hold under CH?

If yes, MBC (i.e.,  $s_0^* = [2^\omega]^{\leq \aleph_0}$ ) follows from CH (Con(MBC+CH), resp.).

## Theorem

Let  $\{ \mathcal{I}_\alpha : \alpha < \omega_1 \}$  be an  $\aleph_1$ -sized family of Sacks dense ideals. Then there exists an uncountable set  $Y \in \bigcap_{\alpha \in \omega_1} \mathcal{I}_\alpha$ .

Moreover, we can construct the set  $Y$  in such a way that  $Y \notin \mathcal{J}$  for some other Sacks dense ideal  $\mathcal{J}$  (proved(?) 5 days ago (using  $s_0^{\text{trans}}$ )).

$$Y \in s_0 \quad :\iff \quad \forall p \exists q \leq p \quad |[q] \cap Y| \leq \aleph_0$$

## Definition

$$Y \in s_0^{\text{trans}} \quad :\iff \quad \forall p \exists q \leq p \quad \forall t \in 2^\omega \quad |(t + [q]) \cap Y| \leq \aleph_0$$

## Question

What can we say about the family  $s_0^{\text{trans}}$ ? Any relation to null-additive?

## Question

Does  $[2^\omega]^{\leq \aleph_0} = \bigcap \{ \mathcal{I} : \mathcal{I} \text{ is S.d.i.} \}$  (at least consistently) hold under CH?

If yes, MBC (i.e.,  $s_0^* = [2^\omega]^{\leq \aleph_0}$ ) follows from CH (Con(MBC+CH), resp.).

## Theorem

Let  $\{ \mathcal{I}_\alpha : \alpha < \omega_1 \}$  be an  $\aleph_1$ -sized family of Sacks dense ideals. Then there exists an uncountable set  $Y \in \bigcap_{\alpha < \omega_1} \mathcal{I}_\alpha$ .

Moreover, we can construct the set  $Y$  in such a way that  $Y \notin \mathcal{J}$  for some other Sacks dense ideal  $\mathcal{J}$  (proved(?) 5 days ago (using  $s_0^{\text{trans}}$ )).

$$Y \in s_0 \quad :\iff \quad \forall p \exists q \leq p \quad |[q] \cap Y| \leq \aleph_0$$

## Definition

$$Y \in s_0^{\text{trans}} \quad :\iff \quad \forall p \exists q \leq p \quad \forall t \in 2^\omega \quad |(t + [q]) \cap Y| \leq \aleph_0$$

## Question

What can we say about the family  $s_0^{\text{trans}}$ ? Any relation to null-additive?

## Question

Does  $[2^\omega]^{\leq \aleph_0} = \bigcap \{ \mathcal{I} : \mathcal{I} \text{ is S.d.i.} \}$  (at least consistently) hold under CH?

If yes, MBC (i.e.,  $s_0^* = [2^\omega]^{\leq \aleph_0}$ ) follows from CH (Con(MBC+CH), resp.).

## Theorem

Let  $\{ \mathcal{I}_\alpha : \alpha < \omega_1 \}$  be an  $\aleph_1$ -sized family of Sacks dense ideals. Then there exists an uncountable set  $Y \in \bigcap_{\alpha \in \omega_1} \mathcal{I}_\alpha$ .

Moreover, we can construct the set  $Y$  in such a way that  $Y \notin \mathcal{J}$  for some other Sacks dense ideal  $\mathcal{J}$  (proved(?) 5 days ago (using  $s_0^{\text{trans}}$ )).

$$Y \in s_0 \quad :\iff \quad \forall p \exists q \leq p \quad |[q] \cap Y| \leq \aleph_0$$

## Definition

$$Y \in s_0^{\text{trans}} \quad :\iff \quad \forall p \exists q \leq p \quad \forall t \in 2^\omega \quad |(t + [q]) \cap Y| \leq \aleph_0$$

## Question

What can we say about the family  $s_0^{\text{trans}}$ ? Any relation to null-additive?

## Question

Does  $[2^\omega]^{\leq \aleph_0} = \bigcap \{ \mathcal{I} : \mathcal{I} \text{ is S.d.i.} \}$  (at least consistently) hold under CH?

If yes, MBC (i.e.,  $s_0^* = [2^\omega]^{\leq \aleph_0}$ ) follows from CH (Con(MBC+CH), resp.).

## Theorem

Let  $\{ \mathcal{I}_\alpha : \alpha < \omega_1 \}$  be an  $\aleph_1$ -sized family of Sacks dense ideals. Then there exists an uncountable set  $Y \in \bigcap_{\alpha < \omega_1} \mathcal{I}_\alpha$ .

Moreover, we can construct the set  $Y$  in such a way that  $Y \notin \mathcal{J}$  for some other Sacks dense ideal  $\mathcal{J}$  (proved(?) 5 days ago (using  $s_0^{\text{trans}}$ )).

$$Y \in s_0 \quad :\iff \quad \forall p \exists q \leq p \quad |[q] \cap Y| \leq \aleph_0$$

## Definition

$$Y \in s_0^{\text{trans}} \quad :\iff \quad \forall p \exists q \leq p \quad \forall t \in 2^\omega \quad |(t + [q]) \cap Y| \leq \aleph_0$$

## Question

What can we say about the family  $s_0^{\text{trans}}$ ? Any relation to null-additive?

## Question

Does  $[2^\omega]^{\leq \aleph_0} = \bigcap \{ \mathcal{I} : \mathcal{I} \text{ is S.d.i.} \}$  (at least consistently) hold under CH?

If yes, MBC (i.e.,  $s_0^* = [2^\omega]^{\leq \aleph_0}$ ) follows from CH (Con(MBC+CH), resp.).

## Theorem

Let  $\{ \mathcal{I}_\alpha : \alpha < \omega_1 \}$  be an  $\aleph_1$ -sized family of Sacks dense ideals. Then there exists an uncountable set  $Y \in \bigcap_{\alpha < \omega_1} \mathcal{I}_\alpha$ .

Moreover, we can construct the set  $Y$  in such a way that  $Y \notin \mathcal{J}$  for some other Sacks dense ideal  $\mathcal{J}$  (proved(?) 5 days ago (using  $s_0^{\text{trans}}$ )).

$$Y \in s_0 \quad :\iff \quad \forall p \exists q \leq p \quad |[q] \cap Y| \leq \aleph_0$$

## Definition

$$Y \in s_0^{\text{trans}} \quad :\iff \quad \forall p \exists q \leq p \quad \forall t \in 2^\omega \quad |(t + [q]) \cap Y| \leq \aleph_0$$

## Question

What can we say about the family  $s_0^{\text{trans}}$ ? Any relation to null-additive?



# References



Timothy J. Carlson.

Strong measure zero and strongly meager sets.

*Proc. Amer. Math. Soc.*, 118(2):577–586, 1993.



Martin Goldstern, Jakob Kellner, Saharon Shelah, and Wolfgang Wohofsky.

Borel Conjecture and dual Borel Conjecture.

*Transactions of the American Mathematical Society*, to appear.

<http://arxiv.org/abs/1105.0823>



Richard Laver.

On the consistency of Borel's conjecture.

*Acta Math.*, 137:151–169, 1976.



Janusz Pawlikowski.

A characterization of strong measure zero sets.

*Israel J. Math.*, 93:171–183, 1996.

My website: <http://wohofsky.eu/math/>

Thank you for your attention and enjoy Warsaw...



Myself in Wrocław

Thank you for your attention and enjoy Warsaw...



Danube in Vienna