Variants of the Borel Conjecture and Sacks dense ideals

Wolfgang Wohofsky

Vienna University of Technology (TU Wien)
and
Kurt Gödel Research Center, Vienna (KGRC)

wolfgang.wohofsky@gmx.at

Trends in set theory
Warsaw, Poland, July 08-11, 2012

Outline of the talk

1. Special sets of real numbers, Borel Conjecture
   - strong measure zero, strongly meager
   - Borel Conjecture, dual Borel Conjecture, Con(BC + dBC)
2. Another variant of the Borel Conjecture
   - Marczewski ideal s_0, “Marczewski Borel Conjecture”
   - ... investigating “Sacks dense ideals”

Outline of the talk

1. Special sets of real numbers, Borel Conjecture
   - strong measure zero, strongly meager
   - Borel Conjecture, dual Borel Conjecture, Con(BC + dBC)
2. Another variant of the Borel Conjecture
   - Marczewski ideal s_0, "Marczewski Borel Conjecture"
   - ...investigating "Sacks dense ideals"

Special sets of real numbers, Borel Conjecture

1. **Special sets of real numbers, Borel Conjecture**
   - strong measure zero, strongly meager
   - Borel Conjecture, dual Borel Conjecture, Con(BC + dBC)
2. Another variant of the Borel Conjecture
   - Marczewski ideal s_0, "Marczewski Borel Conjecture"
   - ... investigating "Sacks dense ideals"

The real numbers: topology, measure, algebraic structure

The real numbers ("the reals")

- $\mathbb{R}$, the classical real line
- 2^ω, the Cantor space (totally disconnected, compact)

Structure on the reals:

- natural topology (intervals/basic clopen sets form a basis)
- standard (Lebesgue) measure
- group structure
  - $(2^\omega,+)$ is a topological group, with $+$ bitwise modulo 2
- Two translation-invariant σ-ideals
  - meager sets $\mathcal{M}$
  - measure zero sets $\mathcal{N}$

The real numbers: topology, measure, algebraic structure

The real numbers ("the reals")

- $\mathbb{R}$, the classical real line
- 2^ω, the Cantor space (totally disconnected, compact)

Structure on the reals:

- natural topology (intervals/basic clopen sets form a basis)
- standard (Lebesgue) measure
- group structure
  - $(2^\omega, +)$ is a topological group, with $+$ bitwise modulo 2
- Two translation-invariant σ-ideals
  - meager sets $\mathcal{M}$
  - measure zero sets $\mathcal{N}$

The real numbers: topology, measure, algebraic structure

The real numbers ("the reals")

- $\mathbb{R}$, the classical real line
- 2^ω, the Cantor space (totally disconnected, compact)

Structure on the reals:

- natural topology (intervals/basic clopen sets form a basis)
- standard (Lebesgue) measure
- group structure
  - $(2^\omega,+)$ is a topological group, with $+$ bitwise modulo 2
- Two translation-invariant σ-ideals
  - meager sets $\mathcal{M}$
  - measure zero sets $\mathcal{N}$

Strong measure zero sets

For an interval $I \subseteq \mathbb{R}$, let $\lambda(I)$ denote its length.

Definition (well-known)

A set $X \subseteq \mathbb{R}$ is (Lebesgue) measure zero ($X \in \mathcal{N}$) iff
for each positive real number $\varepsilon > 0$
there is a sequence of intervals $(I_n)_{n<\omega}$ of total length $\sum_{n<\omega} \lambda(I_n) \leq \varepsilon$
such that $X \subseteq \bigcup_{n<\omega} I_n$.

Definition (Borel; 1919)

A set $X \subseteq \mathbb{R}$ is strong measure zero ($X \in \mathcal{SN}$) iff
for each sequence of positive real numbers $(\varepsilon_n)_{n<\omega}$
there is a sequence of intervals $(I_n)_{n<\omega}$ with $\forall n \in \omega \ \lambda(I_n) \leq \varepsilon_n$
such that $X \subseteq \bigcup_{n<\omega} I_n$.

Strong measure zero sets

For an interval $I \subseteq \mathbb{R}$, let $\lambda(I)$ denote its length.

Definition (well-known)

A set $X \subseteq \mathbb{R}$ is (Lebesgue) measure zero ($X \in \mathcal{N}$) iff
for each positive real number $\varepsilon > 0$
there is a sequence of intervals $(I_n)_{n<\omega}$ of total length $\sum_{n<\omega} \lambda(I_n) \leq \varepsilon$
such that $X \subseteq \bigcup_{n<\omega} I_n$.

Definition (Borel; 1919)

A set $X \subseteq \mathbb{R}$ is strong measure zero ($X \in \mathcal{SN}$) iff
for each sequence of positive real numbers $(\varepsilon_n)_{n<\omega}$
there is a sequence of intervals $(I_n)_{n<\omega}$ with $\forall n \in \omega \ \lambda(I_n) \leq \varepsilon_n$
such that $X \subseteq \bigcup_{n<\omega} I_n$.

Equivalent characterization of strong measure zero sets

For $Y, Z \subseteq 2^\omega$, let $Y + Z = \{y + z : y \in Y, z \in Z\}$.

Key Theorem (Galvin,Mycielski,Solovay; 1973)

A set $Y \subseteq 2^\omega$ is strong measure zero if and only if for every meager set $M \in \mathcal{M}$, $Y + M \neq 2^\omega$.

Note that $Y + M \neq 2^\omega$ if and only if Y can be "translated away" from M, i.e., there exists a $t \in 2^\omega$ such that $(Y + t) \cap M = \emptyset$.

Key Definition

Let $\mathcal{J} \subseteq \mathcal{P}(2^\omega)$ be arbitrary. Define

$$\mathcal{J}^\star := \{Y \subseteq 2^\omega : Y + Z \neq 2^\omega \text{ for every set } Z \in \mathcal{J}\}.$$

$\mathcal{J}^\star$ is the collection of "$\mathcal{J}$-shiftable sets", i.e., $Y \in \mathcal{J}^\star$ iff Y can be translated away from every set in $\mathcal{J}$.

Equivalent characterization of strong measure zero sets

For $Y, Z \subseteq 2^\omega$, let $Y + Z = \{y + z : y \in Y, z \in Z\}$.

Key Theorem (Galvin, Mycielski, Solovay; 1973)

A set $Y \subseteq 2^\omega$ is strong measure zero if and only if for every meager set $M \in \mathcal{M}$, $Y + M \neq 2^\omega$.

Note that $Y + M \neq 2^\omega$ if and only if Y can be "translated away" from M, i.e., there exists a $t \in 2^\omega$ such that $(Y + t) \cap M = \emptyset$.

Key Definition

Let $\mathcal{J} \subseteq \mathcal{P}(2^\omega)$ be arbitrary. Define

$$\mathcal{J}^\star := \{Y \subseteq 2^\omega : Y + Z \neq 2^\omega \text{ for every set } Z \in \mathcal{J}\}.$$

$\mathcal{J}^\star$ is the collection of "$\mathcal{J}$-shiftable sets", i.e., $Y \in \mathcal{J}^\star$ iff Y can be translated away from every set in $\mathcal{J}$.

Strongly meager sets

Key Definition (from previous slide)

Let $\mathcal{J} \subseteq \mathcal{P}(2^\omega)$ be arbitrary. Define

$$\mathcal{J}^\star := \{Y \subseteq 2^\omega : Y + Z \neq 2^\omega \text{ for every set } Z \in \mathcal{J}\}.$$

Key Theorem (Galvin,Mycielski,Solovay; 1973)

A set Y is strong measure zero if and only if it is "$\mathcal{M}$-shiftable", i.e.,

$$\mathcal{SN} = \mathcal{M}^\star$$

Replacing $\mathcal{M}$ by $\mathcal{N}$ yields a notion *dual to strong measure zero*:

Definition

A set Y is strongly meager ($Y \in \mathcal{SM}$) iff it is "$\mathcal{N}$-shiftable", i.e.,

$$\mathcal{SM} := \mathcal{N}^\star$$

Strongly meager sets

Key Definition (from previous slide)

Let $\mathcal{J} \subseteq \mathcal{P}(2^\omega)$ be arbitrary. Define

$$\mathcal{J}^\star := \{Y \subseteq 2^\omega : Y + Z \neq 2^\omega \text{ for every set } Z \in \mathcal{J}\}.$$

Key Theorem (Galvin,Mycielski,Solovay; 1973)

A set Y is strong measure zero if and only if it is "$\mathcal{M}$-shiftable", i.e.,

$$\mathcal{SN} = \mathcal{M}^\star$$

Replacing $\mathcal{M}$ by $\mathcal{N}$ yields a notion *dual to strong measure zero*:

Definition

A set Y is strongly meager ($Y \in \mathcal{SM}$) iff it is "$\mathcal{N}$-shiftable", i.e.,

$$\mathcal{SM} := \mathcal{N}^\star$$

Strongly meager sets

Key Definition (from previous slide)

Let $\mathcal{J} \subseteq \mathcal{P}(2^\omega)$ be arbitrary. Define

$$\mathcal{J}^\star := \{Y \subseteq 2^\omega : Y + Z \neq 2^\omega \text{ for every set } Z \in \mathcal{J}\}.$$

Key Theorem (Galvin,Mycielski,Solovay; 1973)

A set Y is strong measure zero if and only if it is "$\mathcal{M}$-shiftable", i.e.,

$$\mathcal{SN} = \mathcal{M}^\star$$

Replacing $\mathcal{M}$ by $\mathcal{N}$ yields a notion *dual to strong measure zero*:

Definition

A set Y is strongly meager ($Y \in \mathcal{SM}$) iff it is "$\mathcal{N}$-shiftable", i.e.,

$$\mathcal{SM} := \mathcal{N}^\star$$

Borel Conjecture + dual Borel Conjecture

Definition

The Borel Conjecture (BC) is the statement that there are **no** uncountable strong measure zero sets, i.e., $\mathcal{SN} = \mathcal{M}^\star = [2^\omega]^{\leq\aleph_0}$.

- Con(BC), actually BC holds in the Laver model (Laver, 1976)

Definition

The dual Borel Conjecture (dBC) is the statement that there are **no** uncountable strongly meager sets, i.e., $\mathcal{SM} = \mathcal{N}^\star = [2^\omega]^{\leq\aleph_0}$.

- Con(dBC), actually dBC holds in the Cohen model (Carlson, 1993)

Theorem (Goldstern,Kellner,Shelah,W.; 2011)

There is a model of ZFC in which both the Borel Conjecture and the dual Borel Conjecture hold, i.e., Con(BC + dBC).

Borel Conjecture + dual Borel Conjecture

Definition

The Borel Conjecture (BC) is the statement that there are **no** uncountable strong measure zero sets, i.e., $\mathcal{SN} = \mathcal{M}^\star = [2^\omega]^{\leq\aleph_0}$.

- Con(BC), actually BC holds in the Laver model (Laver, 1976)

Definition

The dual Borel Conjecture (dBC) is the statement that there are **no** uncountable strongly meager sets, i.e., $\mathcal{SM} = \mathcal{N}^\star = [2^\omega]^{\leq\aleph_0}$.

- Con(dBC), actually dBC holds in the Cohen model (Carlson, 1993)

Theorem (Goldstern,Kellner,Shelah,W.; 2011)

There is a model of ZFC in which both the Borel Conjecture and the dual Borel Conjecture hold, i.e., Con(BC + dBC).

Borel Conjecture + dual Borel Conjecture

Definition

The Borel Conjecture (BC) is the statement that there are **no** uncountable strong measure zero sets, i.e., $\mathcal{SN} = \mathcal{M}^\star = [2^\omega]^{\leq\aleph_0}$.

- Con(BC), actually BC holds in the Laver model (Laver, 1976)

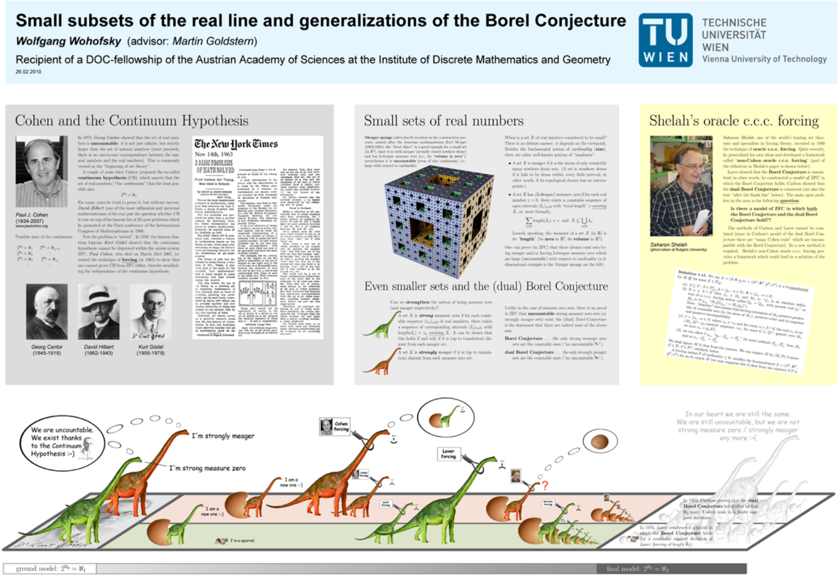
Definition

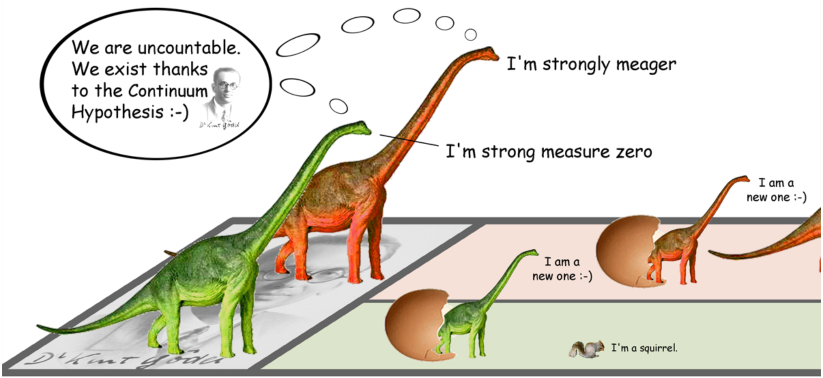
The dual Borel Conjecture (dBC) is the statement that there are **no** uncountable strongly meager sets, i.e., $\mathcal{SM} = \mathcal{N}^\star = [2^\omega]^{\leq\aleph_0}$.

- Con(dBC), actually dBC holds in the Cohen model (Carlson, 1993)

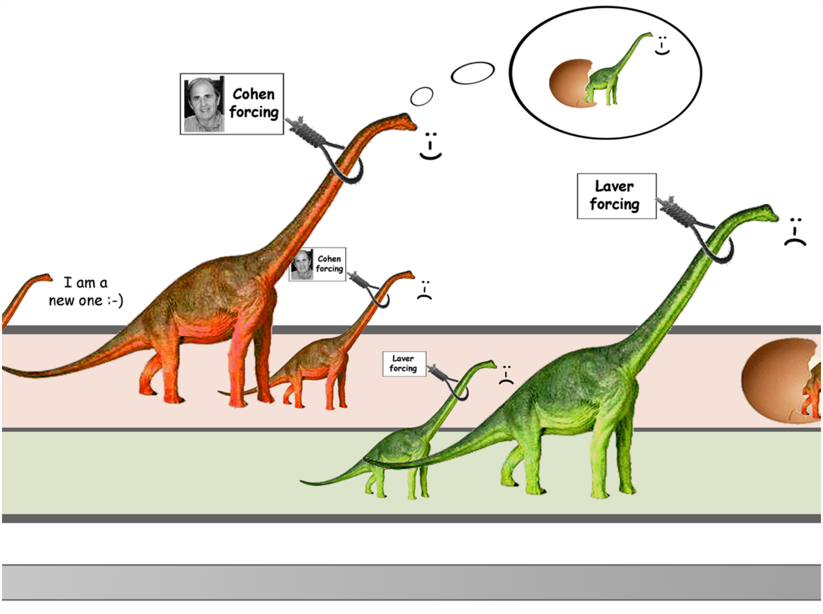
Theorem (Goldstern,Kellner,Shelah,W.; 2011)

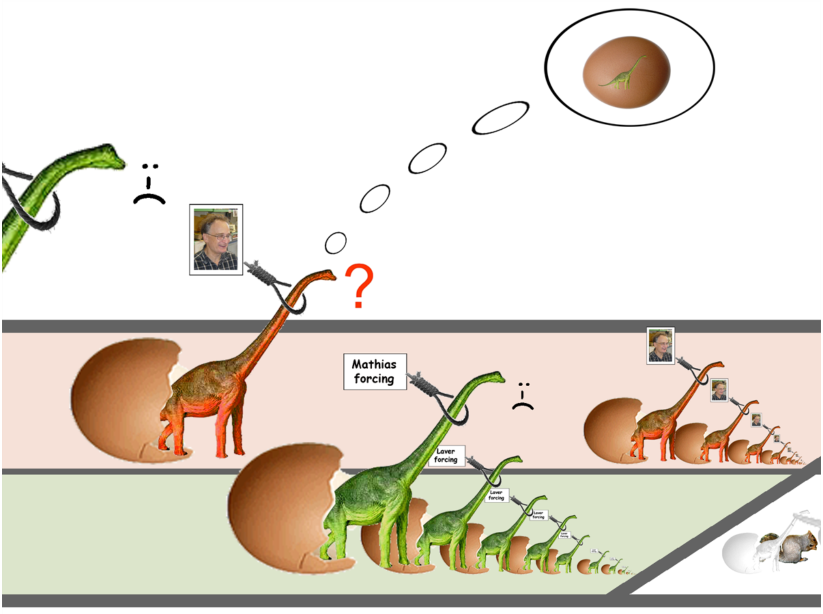
There is a model of ZFC in which both the Borel Conjecture and the dual Borel Conjecture hold, i.e., Con(BC + dBC).


Small subsets of the real line and generalizations of the Borel Conjecture
Wolfgang Wohofsky (advisor: Martin Goldstern)
Recipient of a DOC-fellowship of the Austrian Academy of Sciences at the Institute of Discrete Mathematics and Geometry
26.02.2010
TECHNISCHE UNIVERSITÄT WIEN
Vienna University of Technology
Cohen and the Continuum Hypothesis
Paul J. Cohen (1934-2007)
The New York Times
Georg Cantor (1845-1918)
David Hilbert (1862-1943)
Kurt Gödel (1906-1978)
Small sets of real numbers
Even smaller sets and the (dual) Borel Conjecture
Shelah's oracle c.c.c. forcing
Saharon Shelah
We are uncountable. We exist thanks to the Continuum Hypothesis :-)
I'm strongly meager
I'm strong measure zero
Cohen forcing
Laver forcing
In our heart we are still the same. We are still uncountable, but we are not strong measure zero / strongly meager any more :-(
ground model: $2^{\aleph_0} = \aleph_1$
final model: $2^{\aleph_0} = \aleph_2$



ground model: $2^{\aleph_0} = \aleph_1$


Cohen forcing
Laver forcing
Cohen forcing
I am a new one :-)
Laver forcing


?
Mathias forcing
Laver forcing

Another variant of the Borel Conjecture

1. Special sets of real numbers, Borel Conjecture
   - strong measure zero, strongly meager
   - Borel Conjecture, dual Borel Conjecture, Con(BC + dBC)
2. **Another variant of the Borel Conjecture**
   - Marczewski ideal s_0, "Marczewski Borel Conjecture"
   - . . . investigating "Sacks dense ideals"

Marczewski Borel Conjecture (MBC)

Assume that $\mathcal{J} \subseteq \mathcal{P}(2^\omega)$ is a translation-invariant σ-ideal. Recall that

$$\mathcal{J}^\star := \{Y \subseteq 2^\omega : Y + Z \neq 2^\omega \text{ for every set } Z \in \mathcal{J}\}.$$

Definition

The $\mathcal{J}$-Borel Conjecture ($\mathcal{J}$-BC) the statement that there are **no** uncountable $\mathcal{J}$-shiftable sets, i.e., $\mathcal{J}^\star = [2^\omega]^{\leq\omega}$.

The Marczewski ideal s_0 is the collection of all $Z \subseteq 2^\omega$ such that for each perfect set P, there exists a perfect subset $Q \subseteq P$ with $Q \cap Z = \emptyset$.

Definition

The Marczewski Borel Conjecture (MBC) is the statement that there are **no** uncountable s_0-shiftable sets, i.e., ${s_0}^\star = [2^\omega]^{\leq\omega}$.

What about Con(MBC)?

Can MBC be forced?

Marczewski Borel Conjecture (MBC)

Assume that $\mathcal{J} \subseteq \mathcal{P}(2^\omega)$ is a translation-invariant σ-ideal. Recall that

$$\mathcal{J}^\star := \{Y \subseteq 2^\omega : Y + Z \neq 2^\omega \text{ for every set } Z \in \mathcal{J}\}.$$

Definition

The $\mathcal{J}$-Borel Conjecture ($\mathcal{J}$-BC) the statement that there are **no** uncountable $\mathcal{J}$-shiftable sets, i.e., $\mathcal{J}^\star = [2^\omega]^{\leq\omega}$.

The Marczewski ideal s_0 is the collection of all $Z \subseteq 2^\omega$ such that for each perfect set P, there exists a perfect subset $Q \subseteq P$ with $Q \cap Z = \emptyset$.

Definition

The Marczewski Borel Conjecture (MBC) is the statement that there are **no** uncountable s_0-shiftable sets, i.e., ${s_0}^\star = [2^\omega]^{\leq\omega}$.

What about Con(MBC)?

Can MBC be forced?

Marczewski Borel Conjecture (MBC)

Assume that $\mathcal{J} \subseteq \mathcal{P}(2^\omega)$ is a translation-invariant σ-ideal. Recall that

$$\mathcal{J}^\star := \{Y \subseteq 2^\omega : Y + Z \neq 2^\omega \text{ for every set } Z \in \mathcal{J}\}.$$

Definition

The $\mathcal{J}$-Borel Conjecture ($\mathcal{J}$-BC) the statement that there are **no** uncountable $\mathcal{J}$-shiftable sets, i.e., $\mathcal{J}^\star = [2^\omega]^{\leq\omega}$.

The Marczewski ideal s_0 is the collection of all $Z \subseteq 2^\omega$ such that for each perfect set P, there exists a perfect subset $Q \subseteq P$ with $Q \cap Z = \emptyset$.

Definition

The Marczewski Borel Conjecture (MBC) is the statement that there are **no** uncountable s_0-shiftable sets, i.e., $s_0{}^\star = [2^\omega]^{\leq\omega}$.

What about Con(MBC)?

Can MBC be forced?

Marczewski Borel Conjecture (MBC)

Assume that $\mathcal{J} \subseteq \mathcal{P}(2^\omega)$ is a translation-invariant σ-ideal. Recall that

$$\mathcal{J}^\star := \{Y \subseteq 2^\omega : Y + Z \neq 2^\omega \text{ for every set } Z \in \mathcal{J}\}.$$

Definition

The $\mathcal{J}$-Borel Conjecture ($\mathcal{J}$-BC) the statement that there are **no** uncountable $\mathcal{J}$-shiftable sets, i.e., $\mathcal{J}^\star = [2^\omega]^{\leq\omega}$.

The Marczewski ideal s_0 is the collection of all $Z \subseteq 2^\omega$ such that for each perfect set P, there exists a perfect subset $Q \subseteq P$ with $Q \cap Z = \emptyset$.

Definition

The Marczewski Borel Conjecture (MBC) is the statement that there are **no** uncountable s_0-shiftable sets, i.e., ${s_0}^\star = [2^\omega]^{\leq\omega}$.

What about Con(MBC)?

Can MBC be forced?

Marczewski Borel Conjecture (MBC)

Assume that $\mathcal{J} \subseteq \mathcal{P}(2^\omega)$ is a translation-invariant σ-ideal. Recall that

$$\mathcal{J}^\star := \{Y \subseteq 2^\omega : Y + Z \neq 2^\omega \text{ for every set } Z \in \mathcal{J}\}.$$

Definition

The $\mathcal{J}$-Borel Conjecture ($\mathcal{J}$-BC) the statement that there are **no** uncountable $\mathcal{J}$-shiftable sets, i.e., $\mathcal{J}^\star = [2^\omega]^{\leq\omega}$.

The Marczewski ideal s_0 is the collection of all $Z \subseteq 2^\omega$ such that for each perfect set P, there exists a perfect subset $Q \subseteq P$ with $Q \cap Z = \emptyset$.

Definition

The Marczewski Borel Conjecture (MBC) is the statement that there are **no** uncountable s_0-shiftable sets, i.e., ${s_0}^\star = [2^\omega]^{\leq\omega}$.

What about Con(MBC)?

Can MBC be forced?


MBC

Sacks dense ideals

Unlike BC and dBC, the status of MBC under CH is unclear...

- Is MBC (i.e., ${s_0}^\star = [2^\omega]^{\leq\aleph_0}$) consistent with CH?
- Or does CH even imply MBC?

To investigate the situation under CH, I introduced the following notion:

Definition

A collection $\mathcal{I} \subseteq \mathcal{P}(2^\omega)$ is a Sacks dense ideal (S.d.i.) iff

- $\mathcal{I}$ is a σ-ideal
- $\mathcal{I}$ is *translation-invariant*
- $\mathcal{I}$ is dense in Sacks forcing, more explicitly, for each perfect $P \subseteq 2^\omega$, there is a perfect subset Q in the ideal, i.e., $\exists Q \subseteq P$, $Q \in \mathcal{I}$

Lemma ("Main Lemma")

Assume CH. Let $\mathcal{I}$ be a Sacks dense ideal. Then ${s_0}^\star \subseteq \mathcal{I}$.

Sacks dense ideals

Unlike BC and dBC, the status of MBC under CH is unclear...

- Is MBC (i.e., ${s_0}^\star = [2^\omega]^{\leq\aleph_0}$) consistent with CH?
- Or does CH even imply MBC?

To investigate the situation under CH, I introduced the following notion:

Definition

A collection $\mathcal{I} \subseteq \mathcal{P}(2^\omega)$ is a Sacks dense ideal (S.d.i.) iff

- $\mathcal{I}$ is a σ-ideal
- $\mathcal{I}$ is *translation-invariant*
- $\mathcal{I}$ is dense in Sacks forcing, more explicitly, for each perfect $P \subseteq 2^\omega$, there is a perfect subset Q in the ideal, i.e., $\exists Q \subseteq P$, $Q \in \mathcal{I}$

Lemma ("Main Lemma")

Assume CH. Let $\mathcal{I}$ be a Sacks dense ideal. Then ${s_0}^\star \subseteq \mathcal{I}$.

Sacks dense ideals

Unlike BC and dBC, the status of MBC under CH is unclear...

- Is MBC (i.e., ${s_0}^\star = [2^\omega]^{\leq\aleph_0}$) consistent with CH?
- Or does CH even imply MBC?

To investigate the situation under CH, I introduced the following notion:

Definition

A collection $\mathcal{I} \subseteq \mathcal{P}(2^\omega)$ is a Sacks dense ideal (S.d.i.) iff

- $\mathcal{I}$ is a σ-ideal
- $\mathcal{I}$ is *translation-invariant*
- $\mathcal{I}$ is dense in Sacks forcing, more explicitly, for each perfect $P \subseteq 2^\omega$, there is a perfect subset Q in the ideal, i.e., $\exists Q \subseteq P$, $Q \in \mathcal{I}$

Lemma ("Main Lemma")

Assume CH. Let $\mathcal{I}$ be a Sacks dense ideal. Then ${s_0}^\star \subseteq \mathcal{I}$.

More and more Sacks dense ideals

Lemma ("Main Lemma"; from previous slide)

Assume CH. Let $\mathcal{I}$ be a Sacks dense ideal. Then ${s_0}^\star \subseteq \mathcal{I}$.

In other words: ${s_0}^\star \subseteq \bigcap\{\mathcal{I} : \mathcal{I} \text{ is a Sacks dense ideal}\}$.

Can we (consistently) find many Sacks dense ideals under CH?

$\mathcal{M}$ $\quad$ $\mathcal{N}$ $\quad$ are Sacks dense ideals

$\supsetneq$ $\quad$ $\subsetneq$

$\mathcal{M} \cap \mathcal{N}$

$\cup\!\!\not\,\,$

$\mathcal{E}$

$\mathcal{SN}$ is NOT a Sacks dense ideal, BUT...

More and more Sacks dense ideals

Lemma ("Main Lemma"; from previous slide)

Assume CH. Let $\mathcal{I}$ be a Sacks dense ideal. Then ${s_0}^\star \subseteq \mathcal{I}$.

In other words: ${s_0}^\star \subseteq \bigcap\{\mathcal{I} : \mathcal{I}$ is a Sacks dense ideal$\}$.

Can we (consistently) find many Sacks dense ideals under CH?

$\mathcal{M}$ $\qquad$ $\mathcal{N}$ $\qquad$ are Sacks dense ideals

$$\mathcal{M} \supsetneq \mathcal{M} \cap \mathcal{N} \subsetneq \mathcal{N}$$

$$\mathcal{M} \cap \mathcal{N} \supsetneq \mathcal{E}$$

$\mathcal{SN}$ is NOT a Sacks dense ideal, BUT...

More and more Sacks dense ideals

Lemma ("Main Lemma"; from previous slide)

Assume CH. Let $\mathcal{I}$ be a Sacks dense ideal. Then ${s_0}^\star \subseteq \mathcal{I}$.

In other words: ${s_0}^\star \subseteq \bigcap\{\mathcal{I} : \mathcal{I} \text{ is a Sacks dense ideal}\}$.

Can we (consistently) find many Sacks dense ideals under CH?

$\mathcal{M}$ $\quad$ $\mathcal{N}$ $\quad$ are Sacks dense ideals

$\supsetneq$ $\quad$ $\subsetneq$

$\mathcal{M} \cap \mathcal{N}$

$\cup\!\!\!\!\neq$

$\mathcal{E}$

$\mathcal{SN}$ is NOT a Sacks dense ideal, BUT...

More and more Sacks dense ideals

Lemma ("Main Lemma"; from previous slide)

Assume CH. Let $\mathcal{I}$ be a Sacks dense ideal. Then $s_0{}^\star \subseteq \mathcal{I}$.

In other words: $s_0{}^\star \subseteq \bigcap\{\mathcal{I} : \mathcal{I}$ is a Sacks dense ideal$\}$.

Can we (consistently) find many Sacks dense ideals under CH?

$\mathcal{M}$ $\quad$ $\mathcal{N}$ $\quad$ are Sacks dense ideals

$\mathcal{M} \supsetneq \mathcal{M} \cap \mathcal{N}$, $\quad \mathcal{N} \supsetneq \mathcal{M} \cap \mathcal{N}$

$\mathcal{M} \cap \mathcal{N} \supsetneq \mathcal{E}$

$\mathcal{SN}$ is NOT a Sacks dense ideal, BUT...

More and more Sacks dense ideals

Lemma (“Main Lemma”; from previous slide)

Assume CH. Let $\mathcal{I}$ be a Sacks dense ideal. Then $s_0{}^\star \subseteq \mathcal{I}$.

In other words: $s_0{}^\star \subseteq \bigcap\{\mathcal{I} : \mathcal{I} \text{ is a Sacks dense ideal}\}$.

Can we (consistently) find many Sacks dense ideals under CH?

$\mathcal{M}$ $\quad\quad$ $\mathcal{N}$ $\quad\quad$ are Sacks dense ideals

$\supsetneq$ $\quad$ $\subsetneq$

$\mathcal{M} \cap \mathcal{N}$

$\cup\!\!\neq$

$\mathcal{E}$

$\mathcal{SN}$ is NOT a Sacks dense ideal, BUT…

$$\mathcal{E}$$

$$\cup\!\!\not\parallel$$

$$\bigcap\{\mathcal{I}_f : f \in \omega^\omega\} \quad \subseteq \text{null-additive} \subseteq \mathcal{SN} \cap \mathcal{SM}$$

$$\cup\!\!\not\parallel$$

$$\bigcap\{\mathcal{I}_f : f \in \omega^\omega\} \cap \mathcal{E}_0$$

$$\cup\!|$$

$\exists$ uncount. $Y \in \bigcap\{\mathcal{I}_\alpha : \alpha \in \omega_1\}$, for any $\aleph_1$-sized family of $\mathcal{I}_\alpha$'s

$\cup\!\!\not\parallel \quad \longleftarrow$ proved(?) 5 days ago (using s_0^{trans})

$$\bigcap\{\mathcal{I} : \mathcal{I} \text{ is a Sacks dense ideal}\}$$

$\cup\!| \quad \longleftarrow$ "Main Lemma"

$$s_0{}^\star$$

$$\cup\!|$$

$$[2^\omega]^{\leq\aleph_0}$$

$\mathcal{E}$

$\cup\!\!\nparallel$

$\bigcap\{\mathcal{I}_f : f \in \omega^\omega\}$ $\subseteq$ null-additive $\subseteq \mathcal{SN} \cap \mathcal{SM}$

$\cup\!\!\nparallel$

$\bigcap\{\mathcal{I}_f : f \in \omega^\omega\} \cap \mathcal{E}_0$

$\cup\!|$

$\exists$ uncount. $Y \in \bigcap\{\mathcal{I}_\alpha : \alpha \in \omega_1\}$, for any $\aleph_1$-sized family of $\mathcal{I}_\alpha$'s

$\cup\!\!\nparallel$ $\longleftarrow$ proved(?) 5 days ago (using s_0^{trans})

$\bigcap\{\mathcal{I} : \mathcal{I} \text{ is a Sacks dense ideal}\}$

$\cup\!|$ $\longleftarrow$ "Main Lemma"

$s_0{}^\star$

$\cup\!|$

$[2^\omega]^{\leq\aleph_0}$

$\mathcal{E}$

$\cup\!\!\not\shortparallel$

$\bigcap\{\mathcal{I}_f : f \in \omega^\omega\}$ $\subseteq$ null-additive $\subseteq \mathcal{SN} \cap \mathcal{SM}$

$\cup\!\!\not\shortparallel$

$\bigcap\{\mathcal{I}_f : f \in \omega^\omega\} \cap \mathcal{E}_0$

$\cup|$

$\exists$ uncount. $Y \in \bigcap\{\mathcal{I}_\alpha : \alpha \in \omega_1\}$, for any $\aleph_1$-sized family of $\mathcal{I}_\alpha$'s

$\cup\!\!\not\shortparallel$ $\longleftarrow$ proved(?) 5 days ago (using s_0^{trans})

$\bigcap\{\mathcal{I} : \mathcal{I}$ is a Sacks dense ideal$\}$

$\cup|$ $\longleftarrow$ "Main Lemma"

$s_0^\star$

$\cup|$

$[2^\omega]^{\leq\aleph_0}$

$$\mathcal{E}$$

$$\cup\!\!\not\parallel$$

$$\bigcap\{\mathcal{I}_f : f \in \omega^\omega\} \quad \subseteq \text{null-additive} \subseteq \mathcal{SN} \cap \mathcal{SM}$$

$$\cup\!\!\not\parallel$$

$$\bigcap\{\mathcal{I}_f : f \in \omega^\omega\} \cap \mathcal{E}_0$$

$$\cup\!|$$

$\exists$ uncount. $Y \in \bigcap\{\mathcal{I}_\alpha : \alpha \in \omega_1\}$, for any $\aleph_1$-sized family of $\mathcal{I}_\alpha$'s

$\cup\!\!\not\parallel$ $\longleftarrow$ proved(?) 5 days ago (using s_0^{trans})

$$\bigcap\{\mathcal{I} : \mathcal{I} \text{ is a Sacks dense ideal}\}$$

$\cup\!|$ $\longleftarrow$ "Main Lemma"

$$s_0^\star$$

$$\cup\!|$$

$$[2^\omega]^{\leq\aleph_0}$$

$\mathcal{E}$

$\cup\!\!\not\!\!\parallel$

$\bigcap\{\mathcal{I}_f : f \in \omega^\omega\}$ $\subseteq$ null-additive $\subseteq \mathcal{SN} \cap \mathcal{SM}$

$\cup\!\!\not\!\!\parallel$

$\bigcap\{\mathcal{I}_f : f \in \omega^\omega\} \cap \mathcal{E}_0$

$\cup|$

$\exists$ uncount. $Y \in \bigcap\{\mathcal{I}_\alpha : \alpha \in \omega_1\}$, for any $\aleph_1$-sized family of $\mathcal{I}_\alpha$'s

$\cup\!\!\not\!\!\parallel$ $\longleftarrow$ proved(?) 5 days ago (using s_0^{trans})

$\bigcap\{\mathcal{I} : \mathcal{I}$ is a Sacks dense ideal$\}$

$\cup|$ $\longleftarrow$ "Main Lemma"

$s_0{}^\star$

$\cup|$

$[2^\omega]^{\leq\aleph_0}$

$$\mathcal{E}$$

$$\cup\!\!\not\Vert$$

$$\bigcap\{\mathcal{I}_f : f \in \omega^\omega\} \quad \subseteq \text{null-additive} \subseteq \mathcal{SN} \cap \mathcal{SM}$$

$$\cup\!\!\not\Vert$$

$$\bigcap\{\mathcal{I}_f : f \in \omega^\omega\} \cap \mathcal{E}_0$$

$$\cup\!\mid$$

$$\exists \text{ uncount. } Y \in \bigcap\{\mathcal{I}_\alpha : \alpha \in \omega_1\}, \text{ for any } \aleph_1\text{-sized family of } \mathcal{I}_\alpha\text{'s}$$

$$\cup\!\!\not\Vert \quad \longleftarrow \text{proved(?) 5 days ago (using } s_0^{\text{trans}})$$

$$\bigcap\{\mathcal{I} : \mathcal{I} \text{ is a Sacks dense ideal}\}$$

$$\cup\!\mid \quad \longleftarrow \text{“Main Lemma”}$$

$$s_0{}^\star$$

$$\cup\!\mid$$

$$[2^\omega]^{\leq\aleph_0}$$

Question

Does $[2^\omega]^{\leq\aleph_0} = \bigcap\{\mathcal{I} : \mathcal{I} \text{ is S.d.i.}\}$ (at least consistently) hold under CH?

If yes, MBC (i.e., $s_0{}^\star = [2^\omega]^{\leq\aleph_0}$) follows from CH (Con(MBC+CH), resp.).

Theorem

Let $\{\mathcal{I}_\alpha : \alpha < \omega_1\}$ be an $\aleph_1$-sized family of Sacks dense ideals. Then there exists an uncountable set $Y \in \bigcap_{\alpha\in\omega_1} \mathcal{I}_\alpha$.
Moreover, we can construct the set Y in such a way that $Y \notin \mathcal{J}$ for some other Sacks dense ideal $\mathcal{J}$ (proved(?) 5 days ago (using s_0^{trans})).

$Y \in s_0 \quad :\Longleftrightarrow \quad \forall p\, \exists q \leq p \qquad |[q] \cap Y| \leq \aleph_0$

Definition

$Y \in s_0^{\text{trans}} :\Longleftrightarrow \forall p\, \exists q \leq p \;\; \forall t \in 2^\omega \;\; |(t + [q]) \cap Y| \leq \aleph_0$

Question

What can we say about the family s_0^{trans}? Any relation to null-additive?

Question

Does $[2^\omega]^{\leq\aleph_0} = \bigcap\{\mathcal{I} : \mathcal{I} \text{ is S.d.i.}\}$ (at least consistently) hold under CH?

If yes, MBC (i.e., $s_0{}^\star = [2^\omega]^{\leq\aleph_0}$) follows from CH (Con(MBC+CH), resp.).

Theorem

Let $\{\mathcal{I}_\alpha : \alpha < \omega_1\}$ be an $\aleph_1$-sized family of Sacks dense ideals. Then there exists an uncountable set $Y \in \bigcap_{\alpha\in\omega_1} \mathcal{I}_\alpha$.
Moreover, we can construct the set Y in such a way that $Y \notin \mathcal{J}$ for some other Sacks dense ideal $\mathcal{J}$ (proved(?) 5 days ago (using s_0^{trans})).

$$Y \in s_0 \quad :\Longleftrightarrow \quad \forall p\, \exists q \leq p \qquad |[q] \cap Y| \leq \aleph_0$$

Definition

$$Y \in s_0^{\text{trans}} :\Longleftrightarrow \forall p\, \exists q \leq p\ \forall t \in 2^\omega\ |(t + [q]) \cap Y| \leq \aleph_0$$

Question

What can we say about the family s_0^{trans}? Any relation to null-additive?

Question

Does $[2^\omega]^{\leq\aleph_0} = \bigcap\{\mathcal{I} : \mathcal{I} \text{ is S.d.i.}\}$ (at least consistently) hold under CH?

If yes, MBC (i.e., $s_0{}^\star = [2^\omega]^{\leq\aleph_0}$) follows from CH (Con(MBC+CH), resp.).

Theorem

Let $\{\mathcal{I}_\alpha : \alpha < \omega_1\}$ be an $\aleph_1$-sized family of Sacks dense ideals. Then there exists an uncountable set $Y \in \bigcap_{\alpha\in\omega_1} \mathcal{I}_\alpha$.
Moreover, we can construct the set Y in such a way that $Y \notin \mathcal{J}$ for some other Sacks dense ideal $\mathcal{J}$ (proved(?) 5 days ago (using s_0^{trans})).

$Y \in s_0 \quad :\Longleftrightarrow \forall p\, \exists q \leq p \qquad |[q] \cap Y| \leq \aleph_0$

Definition

$Y \in s_0^{\text{trans}} :\Longleftrightarrow \forall p\, \exists q \leq p\ \forall t \in 2^\omega\ |(t + [q]) \cap Y| \leq \aleph_0$

Question

What can we say about the family s_0^{trans}? Any relation to null-additive?

Question

Does $[2^\omega]^{\leq\aleph_0} = \bigcap\{\mathcal{I} : \mathcal{I} \text{ is S.d.i.}\}$ (at least consistently) hold under CH?

If yes, MBC (i.e., $s_0{}^\star = [2^\omega]^{\leq\aleph_0}$) follows from CH (Con(MBC+CH), resp.).

Theorem

Let $\{\mathcal{I}_\alpha : \alpha < \omega_1\}$ be an $\aleph_1$-sized family of Sacks dense ideals. Then there exists an uncountable set $Y \in \bigcap_{\alpha\in\omega_1} \mathcal{I}_\alpha$.
Moreover, we can construct the set Y in such a way that $Y \notin \mathcal{J}$ for some other Sacks dense ideal $\mathcal{J}$ (proved(?) 5 days ago (using s_0^{trans})).

$Y \in s_0 \quad :\Longleftrightarrow \quad \forall p\, \exists q \leq p \qquad |[q] \cap Y| \leq \aleph_0$

Definition

$Y \in s_0^{\text{trans}} :\Longleftrightarrow \forall p\, \exists q \leq p\ \forall t \in 2^\omega \ |(t + [q]) \cap Y| \leq \aleph_0$

Question

What can we say about the family s_0^{trans}? Any relation to null-additive?

Question

Does $[2^\omega]^{\leq\aleph_0} = \bigcap\{\mathcal{I} : \mathcal{I} \text{ is S.d.i.}\}$ (at least consistently) hold under CH?

If yes, MBC (i.e., $s_0{}^\star = [2^\omega]^{\leq\aleph_0}$) follows from CH (Con(MBC+CH), resp.).

Theorem

Let $\{\mathcal{I}_\alpha : \alpha < \omega_1\}$ be an $\aleph_1$-sized family of Sacks dense ideals. Then there exists an uncountable set $Y \in \bigcap_{\alpha\in\omega_1} \mathcal{I}_\alpha$.
Moreover, we can construct the set Y in such a way that $Y \notin \mathcal{J}$ for some other Sacks dense ideal $\mathcal{J}$ (proved(?) 5 days ago (using s_0^{trans})).

$$Y \in s_0 \quad :\Longleftrightarrow \quad \forall p\, \exists q \leq p \qquad |[q] \cap Y| \leq \aleph_0$$

Definition

$$Y \in s_0^{\text{trans}} :\Longleftrightarrow \forall p\, \exists q \leq p\ \forall t \in 2^\omega \ |(t + [q]) \cap Y| \leq \aleph_0$$

Question

What can we say about the family s_0^{trans}? Any relation to null-additive?

Question

Does $[2^\omega]^{\leq\aleph_0} = \bigcap\{\mathcal{I} : \mathcal{I} \text{ is S.d.i.}\}$ (at least consistently) hold under CH?

If yes, MBC (i.e., $s_0{}^\star = [2^\omega]^{\leq\aleph_0}$) follows from CH (Con(MBC+CH), resp.).

Theorem

Let $\{\mathcal{I}_\alpha : \alpha < \omega_1\}$ be an $\aleph_1$-sized family of Sacks dense ideals. Then there exists an uncountable set $Y \in \bigcap_{\alpha\in\omega_1} \mathcal{I}_\alpha$.
Moreover, we can construct the set Y in such a way that $Y \notin \mathcal{J}$ for some other Sacks dense ideal $\mathcal{J}$ (proved(?) 5 days ago (using s_0^{trans})).

$$Y \in s_0 \quad :\Longleftrightarrow \quad \forall p \, \exists q \leq p \qquad |[q] \cap Y| \leq \aleph_0$$

Definition

$$Y \in s_0^{\text{trans}} :\Longleftrightarrow \forall p \, \exists q \leq p \;\; \forall t \in 2^\omega \;\; |(t + [q]) \cap Y| \leq \aleph_0$$

Question

What can we say about the family s_0^{trans}? Any relation to null-additive?

References

Timothy J. Carlson.
Strong measure zero and strongly meager sets.
Proc. Amer. Math. Soc., 118(2):577–586, 1993.

Martin Goldstern, Jakob Kellner, Saharon Shelah, and Wolfgang Wohofsky.
Borel Conjecture and dual Borel Conjecture.
Transactions of the American Mathematical Society, to appear.
http://arxiv.org/abs/1105.0823

Richard Laver.
On the consistency of Borel's conjecture.
Acta Math., 137:151–169, 1976.

Janusz Pawlikowski.
A characterization of strong measure zero sets.
Israel J. Math., 93:171–183, 1996.

My website: `http://wohofsky.eu/math/`

Thank you for your attention and enjoy Warsaw. . .

Myself in Wrocław

Thank you for your attention and enjoy Warsaw. . .

Danube in Vienna