On Borel sets belonging to every invariant ccc σ-ideal on $2^{\mathbb{N}}$

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Basic definitions

- A *σ-ideal* on an uncountable Polish space X is a family $\mathcal{I} \subseteq \mathcal{P}(X)$ which is closed under taking subsets and countable unions. We assume that $\mathcal{I}$ contains all singletons and every set from $\mathcal{I}$ is covered by a Borel set from $\mathcal{I}$.
- We say that a σ-ideal $\mathcal{I}$ on X is *ccc* if there is no uncountable family of disjoint Borel subsets of X outside $\mathcal{I}$.
- $2^{\mathbb{N}}$ is considered with the coordinatewise addition modulo 2.
- A σ-ideal $\mathcal{I}$ on $2^{\mathbb{N}}$ is *invariant*, if

$$\forall t \in 2^{\mathbb{N}}\ \forall A \subseteq 2^{\mathbb{N}}\ (A \in \mathcal{I} \Rightarrow t + A \in \mathcal{I}).$$

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Examples

- $\mathscr{M}$ – the σ-ideal of meager subsets of $2^{\mathbb{N}}$,
- $\mathcal{N}$ – the σ-ideal of null subsets of $2^{\mathbb{N}}$,
- $\mathscr{M} \cap \mathcal{N}$,
- $\mathcal{N} \otimes \mathscr{M}$ – the Fubini product of $\mathcal{N}$ and $\mathscr{M}$ is the σ-ideal generated by the Borel sets $B \subseteq 2^{\mathbb{N}} \times 2^{\mathbb{N}}$ with $\{x \in 2^{\mathbb{N}} : \; B_x \notin \mathscr{M}\} \in \mathcal{N}$,
- $\mathscr{M} \otimes \mathcal{N}$ – the Fubini product of $\mathscr{M}$ and $\mathcal{N}$,
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Open problem: Is $\mathcal{I}_{ccc}$ equal to the intersection of all invariant ccc σ-ideals on $2^{\mathbb{N}}$? In other words: is it true for a set $A \subseteq 2^{\mathbb{N}}$ that if for every invariant ccc σ-ideal $\mathcal{I}$ on $2^{\mathbb{N}}$ the set A is covered by a Borel member of $\mathcal{I}$, then A is covered by a Borel set which belongs to every invariant ccc σ-ideal on $2^{\mathbb{N}}$?

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A related σ-ideal

$\mathcal{I}_0$ – the σ-ideal on $2^{\mathbb{N}}$ generated by the family of all Borel sets $B \subseteq 2^{\mathbb{N}}$ such that there exists a perfect set $P \subseteq 2^{\mathbb{N}}$ with $(B + x) \cap (B + y) = \emptyset$ for $x, y \in P$ and $x \neq y$.

$\mathcal{I}_0$ was defined by Bukovsky and studied by Balcerzak, Rosłanowski and Shelah, [*Ideals without ccc*, J. Symb. Logic **63** (1998), 128–148] and Solecki, [*A Fubini theorem*, Topology Appl. **154** (2007), 2462–2464].

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Two kinds of questions concerning $\mathcal{I}_{ccc}$

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A related previous work

A σ-ideal $\mathcal{I}$ on a Polish space X has *property (M)* if there is a Borel surjective function $f : X \to 2^{\mathbb{N}}$ such that all fibers of f are not in $\mathcal{I}$.

Theorem (Balcerzak, Rosłanowski and Shelah)

$\mathcal{I}_0$, the σ-ideal generated by Borel sets having perfectly many pairwise disjoint translates, is not ccc. In fact, $\mathcal{I}_0$ has property (M).

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The motivating question behind the work of B-R-S:

- What can be the reasons for failing ccc?

A related question:

- Can we single out a particular property of a Borel subset of $2^{\mathbb{N}}$ that prevents it from being a member of *any* invariant ccc σ-ideal (like having perfectly many disjoint translates) so that the failure of ccc of an invariant σ-ideal $\mathcal{I}$ (even in the strong form of (M)) is always witnessed by an $\mathcal{I}$-positive Borel set with the property under consideration?

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Structural properties of $\mathcal{I}_{ccc}$

Actually, the answer is just: "NO".

Theorem

$\mathcal{I}_{ccc}$, the σ-ideal on $2^{\mathbb{N}}$ generated by Borel sets which belong to every invariant ccc σ-ideal, is not ccc. In fact, $\mathcal{I}_{ccc}$ has property (M).

Since $\mathcal{I}_0 \subseteq \mathcal{I}_{ccc}$, the latter generalizes the B-R-S theorem.

The proof is based on ideas of Solecki, with further simplifications.

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A sketch of proof of "$\mathcal{I}_{ccc}$ is not ccc"

We say that σ-ideals $\mathcal{I}$ and $\mathcal{J}$ on $2^{\mathbb{N}}$ are *orthogonal* if $2^{\mathbb{N}}$ can be partitioned into *Borel* sets $A,\ B$ with $A \in \mathcal{I}$ and $B \in \mathcal{J}$.

To see that $\mathcal{I}_{ccc}$ is not ccc, it is enough to prove that:

- (Based on an idea of Solecki) There exists a collection of continuum many pairwise orthogonal ccc invariant σ-ideals on $2^{\mathbb{N}}$; in fact, each of them is essentially equal (more precisely, Borel isomorphic) to $\mathcal{N} \otimes \mathscr{M}$.
- (A general observation) If $\mathscr{T}$ is an uncountable collection of pairwise orthogonal ccc σ-ideals on $2^{\mathbb{N}}$, then any ccc σ-ideal on $2^{\mathbb{N}}$ is orthogonal to a member of $\mathscr{T}$; in particular, their intersection is not ccc.

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$2^{\aleph_0}$ many pairwise orthogonal ccc invariant σ-ideals on $2^{\mathbb{N}}$

Let $G_y = (2^{\mathbb{N}})^{\{n: y(n)=0\}}$, $H_y = (2^{\mathbb{N}})^{\{n: y(n)=1\}}$; for $y \in 2^{\mathbb{N}} \setminus \{\bar{0}, \bar{1}\}$.

Identifying $2^{\mathbb{N}}$ with $G_y \times H_y$ let $\mathcal{I}_y = \mathcal{N}_y \otimes \mathscr{M}_y$ where $\mathcal{N}_y$ is the σ-ideal of null sets in G_y (with respect to the product of ordinary measures on $2^{\mathbb{N}}$) and $\mathscr{M}_y$ is the σ-ideal of meager subsets of H_y.

Partition $2^{\mathbb{N}}$ into Borel sets: A of measure 1 and comeager B.

Let $C_y = A^{\{n: y(n)=0\}} \times B^{\{n: y(n)=1\}}$.

To see that $\mathcal{I}_y \perp \mathcal{I}_z$ if $y \neq z$, note that $2^{\mathbb{N}} \setminus C_y \in \mathcal{I}_y$, $2^{\mathbb{N}} \setminus C_z \in \mathcal{I}_z$ and $C_y \cap C_z = \emptyset$.

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A byproduct: a new proof of a lower bound for $non(\mathcal{I})$

Proposition (Recław)

Let $\mathcal{I}$ be an invariant ccc σ-ideal on $2^{\mathbb{N}}$. Then

$$non(\mathcal{I}) \geqslant \min(cov(\mathcal{N}), cov(\mathcal{M})).$$

Proof:

- Rothberger: If $\mathcal{I}$ and $\mathcal{J}$ are orthogonal invariant ccc σ-ideals on $2^{\mathbb{N}}$, then $\mathrm{non}(\mathcal{I}) \geqslant \mathrm{cov}(\mathcal{J})$.
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Related results:

- Recław (1998): $\mathrm{non}(\mathcal{I}) \geqslant p$; p=the pseudointersection number,
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$F_\kappa :\ 2^{\mathbb{N}} \times (2^{\mathbb{N}})^\kappa \to (2^{\mathbb{N}})^\kappa$ is the action of $2^{\mathbb{N}}$ on the product group $(2^{\mathbb{N}})^\kappa$ by coordinate-wise translations, i.e.,

$$F_\kappa(t, \langle x_\alpha : \alpha < \kappa \rangle) = \langle t + x_\alpha : \alpha < \kappa \rangle.$$

$IS_\kappa(P)$ is the set of all injective sequences of length κ of elements of $P \subseteq 2^{\mathbb{N}}$.

Lemma

For $0 < \kappa \leqslant \aleph_1$ and $A,\ P \subseteq 2^{\mathbb{N}}$, $|P| \geqslant \aleph_1$, the following are equivalent:

- *P is a witness that A is κ-small,*
- *$F_\kappa[2^{\mathbb{N}} \times A^\kappa] \cap IS_\kappa(P) = \emptyset$.*

Small sets - a related previous work

In the terminology of Kubiś [*Perfect cliques and G_δ colorings of Polish spaces*, Proc. Amer. Math. Soc. **131** (2003), 619–623] and Mátrai [*Infinite dimensional perfect set theorems*, Trans. Amer. Math. Soc., in press]:

- P is a witness that A is κ-small iff P is C*-homogeneous* where C is the complement of $F_\kappa[2^{\mathbb{N}} \times A^\kappa]$ in $(2^{\mathbb{N}})^\kappa$ (i.e., all injective κ-sequences of elements of P are in C).

Results of Kubiś and Matrai assuring (in certain cases) that if there exists an uncountable C-homogeneous set then there exists also a perfect one, readily give:

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Small versus perectly small

Proposition

Let $A \subseteq 2^{\mathbb{N}}$.

- *If A is F_σ and n-small for a certain n, $0 < n < \aleph_0$, then A is perfectly n-small.*
- *If A is analytic and $\aleph_0$-small with a non-meager witness, then A is perfectly $\aleph_0$-small.*
- *Let $\aleph_1 < \lambda \leqslant \mathfrak{c}$. It is consistent with ZFC that if A is analytic and $\aleph_0$-small with a witness of cardinality λ, then A is perfectly $\aleph_0$-small.*

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Open problem: Is every Borel n-small, $1 < n < \aleph_0$, subset of $2^{\mathbb{N}}$ perfectly n-small?

Remark (Matrai)

There exists an F_σ-set $C \subseteq (2^{\mathbb{N}})^2$ with uncountable but not perfect C-homogeneous sets.

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Perfectly 2-small compact sets

Concerning the case $n = 2$, we have

Proposition (Banakh, Lyaskovska and Repovs)

If $C \subseteq 2^{\mathbb{N}}$ is compact, then the following are equivalent:

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Theorem

Let B be a Borel subset of $2^{\mathbb{N}}$.

- *If B is $\aleph_0$-small, then $B \in \mathcal{I}_{ccc}$.*
- *Under $MA + \neg CH$, if B is $\aleph_1$-small, then $B \in \mathcal{I}_{ccc}$.*
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If there exists a Sierpiński set $S \subseteq 2^{\mathbb{N}}$, then every Borel null-set $B \subseteq 2^{\mathbb{N}}$ is $\aleph_1$-small with a witness S. In particular, there is one not in $\mathcal{I}_{ccc}$.

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$\mathcal{I}_{ccc}$ versus perfectly small sets

Open problem:
Is $\mathcal{I}_{ccc}$ generated by Borel perfectly $\aleph_1$-small subsets of $2^{\mathbb{N}}$?

Proposition

- *There exists a compact, perfectly 3-small subset C of $2^{\mathbb{N}}$ such that $C + C = 2^{\mathbb{N}}$. In particular, $C \in \mathcal{I}_{ccc}$ but C is not perfectly 2-small.*
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The second part of the last theorem, i.e.,

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Covering $2^{\mathbb{N}}$ by $< 2^{\aleph_0}$ many translates of a compact set

Observations:

- Assume that $\aleph_1 \leqslant \kappa \leqslant \mathfrak{c}$. Then less than κ many translates of a $\aleph_1$-small set with a witness of cardinality at least κ do not cover $2^{\mathbb{N}}$ (observed by Darji and Keleti)
- It is consistent with the negation of CH that there is a compact null set $A \subseteq 2^{\mathbb{N}}$ such that $A \notin \mathcal{I}_{ccc}$ but no $\aleph_1$ many translates of A cover $2^{\mathbb{N}}$. Clearly, the latter is true for any null set A provided $cov(\mathcal{N}) > \aleph_1$.

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Theorem (Baumgartner)

It is consistent with the negation of CH that $2^{\mathbb{N}}$ can be partitioned into $\aleph_1$ many disjoint non-empty compact sets.

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CH is equivalent to the statement that $2^{\mathbb{N}}$ can be partitioned into $\aleph_1$ many disjoint translates of a compact set.
Moreover, $\neg$CH implies that if $2^{\mathbb{N}}$ is the union of a collection $\mathcal{A}$ of $\aleph_1$ many translates of a compact set then for every natural number n there is $x \in 2^{\mathbb{N}}$ such that x is a member of at least n elements of $\mathcal{A}$.

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A sketch of proof

Proof: Suppose that $A \subseteq 2^{\mathbb{N}}$ is compact and for some $n > 1$ there is a covering of $2^{\mathbb{N}}$ by $\aleph_1$ many translates of A such that no $x \in 2^{\mathbb{N}}$ is a member of n elements of the covering. Then A is n-small and, being compact, it is also perfectly n-small with a perfect witness P. But P being covered by $\aleph_1$ many translates of A and having countable intersection with each of them, this implies that $|P| = \aleph_1$ hence CH holds.

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