THE FREE SET PROPERTY FOR CALIBRATED IDEALS

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Canonization of equivalence relations

Given a Polish space X, a σ-ideal I, and a Borel (or analytic) equivalence relation E,

is there a Borel I-positive set $B \subset X$ such that $E \restriction B$ has a simple form?

Possible outcomes

- Best: $E \restriction B$ is either identity or B^2 (total canonization);

- total canonization for simple equivalences (e.g. classifiable by countable structures);

- canonization up to a known set of obstacles–such as $E \restriction B$ is either identity or B^2 or E_0;

- canonization down to a Borel complexity class–such as $E \restriction B$ is smooth;

- Negative: $E \restriction B$ maintains its complexity on all Borel I-positive sets.

The free set property

Definition. I has the *free set property* if for every analytic I-positive B and every analytic set $D \subset B \times B$ there is a Borel I-positive *free set*, a set B such that $D \cap B \times B$ is a subset of the diagonal.

Example. The meager ideal on 2^ω does not have the free set property. ($D = E_0$)

Example. The σ-ideal generated by compact subsets of ω^ω does have the free set property. (Solecki-Spinas)

Fact. The free set property imples total canonization for analytic equivalence relations.

Calibrated ideals

Definition. A σ-ideal I on a Polish space X is *calibrated* if for every closed I-positive C and closed I-small $D_n : n \in \omega$ there is a closed I-positive $C' \subset C \setminus \bigcup_n D_n$.

Example. The meager ideal is not calibrated–let the sets D_n enumerate a countable dense subset of X.

Example. The ideal of countable sets is calibrated–the set $C \setminus \bigcup_n D_n$ is positive and contains a perfect subset.

Examples of calibrated ideals

Class 1. σ-ideals with *covering property*–every positive analytic set contains a closed positive subset. The ideal of countable sets, the ideal of sets of σ-finite packing measure mass, the ideal of sets of extended uniqueness;

Class 2. σ-ideals obtained from class 1 by taking the subideal σ-generated by closed sets. The σ-ideal generated by closed Lebesgue null sets.

Class 3. Other: the σ-ideal σ-generated by closed sets of uniqueness

Class 4. The σ-ideals with stratified calibration: the σ-ideal generated by closed subsets of $[0,1]^\omega$ of finite dimension.

The main theorem

Theorem. Let I be a σ-ideal on a compact metric space X, σ-generated by a coanalytic collection of compact sets. If I is calibrated, then I has the free set property.

Corollaries for this class of σ-ideals

A. Total canonization for analytic equivalence relations.

B. Silver property for Borel equivalence relations E: either there is a Borel I-positive set of pairwise inequivalent elements, or the whole space decomposes into countably many classes and an I-small set.

C. If Borel E has an I-positive set consisting of pairwise inequivalent elements, then it has a *Borel* such set.

D. The same for finitely many Borel equivalence relations simultaneously.

Canonization of other objects

Example. (I=ideal of countable sets.) If $G \subset 2^\omega \times 2^\omega$ is a graph then there is a perfect set $P \subset 2^\omega$ such that $G \restriction P$ is either $P \times P$ minus the diagonal, or empty.

Example. (I=the σ-ideal generated by closed null sets.) There is a Borel function $f : 2^\omega \times 2^\omega \to 2^\omega$ such that for all Borel I-positive sets B, C, $f''(B \times C) = 2^\omega$.