CHARACTER OF POINTS IN HIGSON CORONA OF A METRIC SPACE

TARAS BANAKH

Given a metric space (X, d_X) denote by $SO(X)$ the linear algebra of bounded slowly oscillating functions on X. A function $f : X \to \mathbb{R}$ is *slowly oscillating* if for each positive real number R there is a bounded subset $B \subset X$ such that for any points $x, y \in X \setminus B$ with $d_X(x, y) \le R$ we get $|f(x) - f(y)| < \frac{1}{R}$. The algebra $SO(X)$ determines the injective map $i : X \to \mathbb{R}^{SO(X)}$, $i : x \mapsto (f(x))_{f \in SO(X)}$ with compact closure $\overline{i(X)} \subset \mathbb{R}^{SO(X)}$.

The *Higson corona* of X is the intersection $\nu(X) = \bigcap_{B \subset X} \overline{i(X \setminus B)}$ where B runs over all bounded subsets of X. The Higson corona $\nu(X)$ is a compact Hausdorff space reflecting some asymptotic properties of X. In particular, its topological dimension $\dim \nu(X)$ is equal to the asymptotic dimension $\mathrm{asdim}(X)$ of X if X is a proper metric space of finite asymptotic dimension, see [**?**]. We recall that a metric space X has *asymptotic dimension* $\mathrm{asdim}(X) \le n$ if for each $\varepsilon > 0$ there is a cover $\mathcal{U}$ of X with finite $\mathrm{mesh}(\mathcal{U}) = \sup_{U \in \mathcal{U}} \mathrm{diam}(U)$ such that each ε-ball $B(x, \varepsilon)$, $x \in X$, meets at most $n + 1$ sets $U \in \mathcal{U}$.

In [**?**] I.V.Protasov proved that under CH, the Higson corona $\nu(X)$ of each unbounded metric space X of asymptotic dimension zero is homeomorphic to the remainder $\omega^* = \beta(\omega) \setminus \omega$ of the Stone-Cech compactification of positive integers and asked if his result remains true in ZFC. We shall give a negative answer to this question calculating the minimal character of points in the Higson corona $\nu(X)$.

By the *character* $\chi(x, X)$ of a point x of a topological space X we understand the smallest cardinality $|\mathcal{B}_x|$ of a neighborhood base $\mathcal{B}_x$ of the topology of X at x. The cardinal $m\chi(X) = \min_{x \in X} \chi(x, X)$ is called the *minimal character* of X. The minimal character $m\chi(\omega^*)$ of the Stone-Cech remainder ω^* is a well-known small uncountable cardinal denoted by $\mathfrak{u}$. By $\mathfrak{d}$ we denote the smallest cardinality of a cover of the Baire space ω^ω by compact subsets. It is knowns that $\mathfrak{u} = \mathfrak{d} = \mathfrak{c}$ under Martin's Axiom, but the strict inequality $\mathfrak{u} < \mathfrak{d}$ is consistent with ZFC.

Our main result is the following:

Theorem 1. *The minimal character $m\chi(\nu(X))$ of the Higson corona $\nu(X)$ of an unbounded metric space X is equal to*

- $\mathfrak{u}$ *if X has asymptotically isolated balls, and*
- $\max\{\mathfrak{u}, \mathfrak{d}\}$*, otherwise.*

We say that a metric space X *has asymptotically isolated balls* if there is $\varepsilon \in (0, \infty)$ such that for each $\delta \in (0, \infty)$ there is a point $x \in X$ such that $B(x, \delta) \subset B(x, \varepsilon)$.

Theorem 1 allows us to give a corona characterization of the Cantor-macro cube $2^{<\mathbb{N}}$. The *Cantor macro-cube* $2^{<\mathbb{N}}$ is the set

$$2^{<\mathbb{N}} = \{(x_n)_{n \in \in} \in \{0, 1\}^{\mathbb{N}} : \exists m \; \forall n \ge m \;\; x_m = 0\}$$

endowed with the metric $d\big((x_n)_{n\in\mathbb{N}},(y_n)_{n\in\mathbb{N}}\big) = \max_{n\in\mathbb{N}} 2^n|x_n - y_n|$. By [**?**], *a metric space X is coarsely equivalent to the Cantor macro-cube $2^{<\mathbb{N}}$ if and only if X is unbounded, has bounded geometry, has asymptotic dimension zero and has no asymptotically isolated balls.*

A metric space X has *bounded geometry* if there is $\varepsilon \in (0,\infty)$ such that for each $\delta \in (0,\infty)$ there is a natural number N such that each δ-ball $B(x,\delta)$, $x \in X$, can be covered by $\le N$ balls of radius ε. Two metric spaces X, Y are *coarsely equivalent* if there are two coarse maps $f : X \to Y$ and $g : Y \to X$ such that $\max\{\sup_{x\in X} d_X(g\circ f(x), x), \sup_{y\in Y} d_Y(f\circ g(y), y)\} < \infty$. A map $f : X \to Y$ between metric spaces is *coarse* if for each $\varepsilon \in (0,\infty)$ there is $\delta \in (0,\infty)$ such that for any two points $x, x' \in X$ with $d_X(x,x') < \varepsilon$ we get $d_Y(f(x), f(x')) < \varepsilon$.

Theorem 2. *Under $(\mathfrak{u} < \mathfrak{d})$ for a metric space X of bounded geometry the following conditions are equivalent:*

(1) *X is coarsely equivalent to the Cantor macro-cube $2^{<\mathbb{N}}$;*
(2) *the Higson coronas $\nu(X)$ and $\nu(2^{<\mathbb{N}})$ are homeomorphic;*
(3) *$\dim(\nu(X)) = 0$ and $m\chi(X) = \mathfrak{d}$.*

The proofs of these theorems can be found in [**?**].

References

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Jan Kochanowski University in Kielce (Poland) and Ivan Franko National University of Lviv (Ukraine)

E-mail address: t.o.banakh@gmail.com