Rapid ultrafilters and summable ideals

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We consider summable ideals

$$\mathcal{I}_g = \{A \subseteq \mathbb{N} : \sum_{a \in A} g(a) < +\infty\}$$

determined by a decreasing function $g : \mathbb{N} \to (0, +\infty)$ with the properties $\sum_{n \in \mathbb{N}} g(n) = +\infty$ and $\lim_{n \to +\infty} g(n) = 0$.

It is known that rapid ultrafilters can be characterized as those ultrafilters which have a nonempty intersection with each tall summable ideal $\mathcal{I}_g$. In fact, the following two statements are equivalent for an ultrafilter $\mathcal{U}$ on the natural numbers:

1. $\mathcal{U} \in \omega^*$ is a rapid ultrafilter
2. for every tall summable ideal $\mathcal{I}_g$ and for every finite-to-one function $f : \omega \to \mathbb{N}$ there exists a set $U \in \mathcal{U}$ such that $f[U] \in \mathcal{I}_g$ (i.e. $\mathcal{U}$ is a weak $\mathcal{I}_g$-ultrafilter)

An ultrafilter $\mathcal{U}$ is called an $\mathcal{I}_g$-ultrafilter for every function $f : \omega \to \mathbb{N}$ there exists a set $U \in \mathcal{U}$ such that $f[U] \in \mathcal{I}_g$.

Obviously, if an ultrafilter $\mathcal{U}$ is an $\mathcal{I}_g$-ultrafilter for all tall summable ideals then $\mathcal{U}$ is rapid. However, if only one summable ideal $\mathcal{I}_g$ is considered, the analogous proposition need not be true. We prove under the assumption of Martin's Axiom for σ-centered posets that for every tall summable ideal $\mathcal{I}_g$ there exists an $\mathcal{I}_g$-ultrafilter which is not rapid.