Definable Hausdorff Gaps

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In 1936, Hausdorff constructed an (ω_1, ω_1)-gap in $\mathcal{P}(\omega)/\text{fin}$, i.e., a pair of sequences $(\{a_\alpha \mid \alpha < \omega_1\}, \{b_\alpha \mid \alpha < \omega_1\})$ well-ordered by $\subseteq^*$, but with no c separating them. In general, we define a *Hausdorff gap* to be any orthogonal pair (A, B), where both A and B are σ-directed with respect to $\subseteq^*$ but there is no c separating A from B.

Stevo Todorcevic [1] proved that there are no analytic Hausdorff gaps. We extend this result by showing that there are no Hausdroff gaps in Solovay's model and assuming $\mathrm{AD}(\mathbb{R})$, and also that the existence of a $\mathbf{\Sigma}^1_2$-definable Hausdorff gap is equivalent to the existence of a $\mathbf{\Pi}^1_1$-definable Hausdorff gap, and equivalent to $\aleph_1$ being inaccessible in L.

[1] Stevo Todorčević, *Analytic gaps*, Fund. Math. 150 (1996)