On a Topological Choice Principle by Murray Bell

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Abstract

In Arnold W. Miller's list of interesting problems ([1, Section 13]), the following question is posed:

Let (C) stand for the statement:

(C) *For every family of non-empty sets there exists a function assigning to each set in the family a compact Hausdorff topology.*

Is (C) *equivalent to the Axiom of Choice* (AC)*? If not, what principles of choice is* (C) *equivalent to*?

In Miller [1], (C) and the related open problem are attributed to Murray Bell. In this paper, we investigate the deductive strength of Bell's principle (C) and its interrelation with various choice forms. Among other results, we prove the following:

(1) The Axiom of Multiple Choice (MC) does not imply (C) in ZFA set theory, i.e., Zermelo-Fraenkel set theory with the Axiom of Extensionality weakened to allow the existence of atoms.

(2) (C) + "There exists a free ultrafilter on ω" implies "For every family $\mathcal{A} = \{A_i : i \in \omega\}$ such that for all $i \in \omega$, $|A_i| \geq 2$, there is an infinite subfamily $\mathcal{B} \subset \mathcal{A}$ and a function f with domain $\mathcal{B}$ such that for all $B \in \mathcal{B}$, $\emptyset \neq f(B) \subsetneqq B$".

(3) (C) + "There exists a free ultrafilter on ω" implies "For every integer $n \geq 2$, every family $\{A_i : i \in \omega\}$ of non-empty sets each of which has at most n elements has an infinite subfamily with a choice function.".

(4) (C) + "Every compact Hausdorff space is effectively normal (i.e., there exists a function F such that for every pair (A, B) of disjoint closed sets in the space, $F(A, B) = (C, D)$, where C and D are disjoint open sets such that $A \subseteq C$ and $B \subseteq D$)" implies MC restricted to families of non-empty sets each expressible as a countable union of finite sets, and "For every integer $n \geq 2$, every family $\{A_i : i \in \omega\}$ of non-empty sets each of which has at most n elements has an infinite subfamily with a choice function".

(5) (C) + "The axiom of choice for countable families of non-empty sets of reals" implies "There exists a non-Lebesgue-measurable set of reals".

(6) (C) + "For every set X, every countable filterbase on X can be extended to an ultrafilter on X" implies $\mathrm{AC}^{\aleph_0}$, i.e., the axiom of choice for countable families of non-empty sets.

(7) (C) restricted to *countable* families of non-empty sets + "For every set X, every countable filterbase on X can be extended to an ultrafilter on X" is equivalent to $\mathrm{AC}^{\aleph_0}$ + "There exists a free ultrafilter on ω".

(8) (C) restricted to countable families of non-empty sets + "For every set X, every countable filterbase on X can be extended to an ultrafilter on X" implies the statements: "The Tychonoff product of a countable family of compact spaces is compact" and "For every infinite set X, the Cantor cube 2^X is countably compact".

(9) (C) restricted to countable families of non-empty sets does not imply "There exists a free ultrafilter on ω" in ZF set theory.

(10) The conjunction of the Countable Union Theorem (The union of a countable family of countable sets is countable) and "Every infinite set is Dedekind-infinite" does not imply (C) restricted to countable families of non-empty sets, in ZFA set theory.

Keywords: Axiom of Choice, weak forms of the Axiom of Choice, compact Hausdorff topological spaces, Fraenkel-Mostowski permutation models.

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References

[1] A. W. Miller, *Some interesting problems*, In Set Theory of the Reals, ed Haim Judah, Israel Mathematical Conference Proceedings, American Math. Society **6** (1993), 645-654. (See also http://www.math.wisc.edu/~miller/res/problem.pdf).

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