Random Walks on Groups

Set up: $\Gamma = \langle S \rangle$, $|S| < +\infty$, $S = S^{-1}$.

GGT: $\rightsquigarrow$ Cay(Γ, S) $\rightsquigarrow$ Geometric prop.

RWT/Ergodic theory $\rightsquigarrow \mu = \frac{1}{S} \sum_{s \in S} \delta_s$

Dirac measure at $s \in S$.

$$\mu = \frac{1}{|S|} \sum_{s \in S} \delta_s$$

WANT TO STUDY:

$$\mu^{*n}(g) = \sum_{g_1 \cdots g_n = g} \mu(g_1) \cdots \mu(g_n)$$

n'th convolution power of μ.

$$\Omega = S^{\mathbb{N}_0} = S \times S \times \ldots$$

$$\mathbb{P}_\mu = \mu^{\mathbb{N}} = \mu \times \mu \times \ldots$$

Given $\omega \in \Omega$ $\quad \omega = (\omega_0, \omega_1, \ldots)$

define:

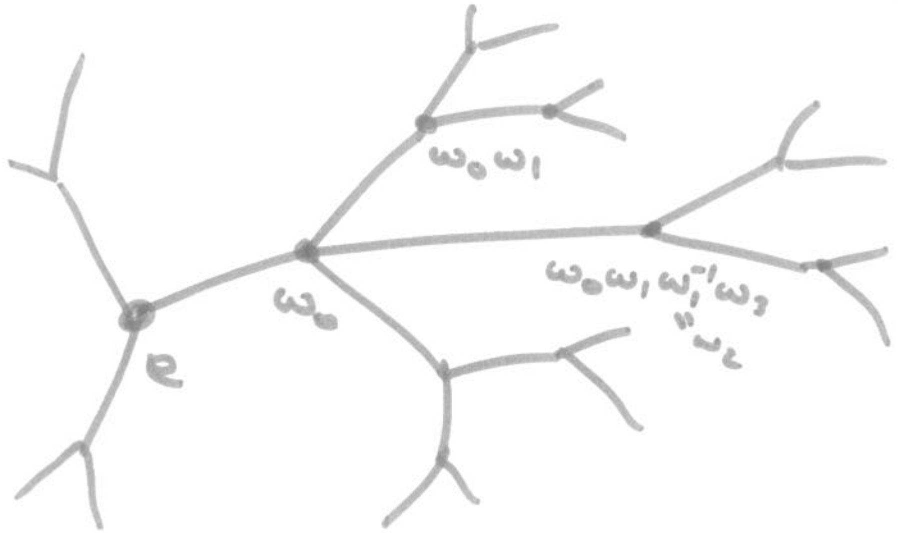
$Z_0(\omega) = e$

$Z_n(\omega) = \omega_0 \omega_1 \cdots \omega_{n-1}\ ,\ n \geq 1$

Simple random walk on (Γ, S).

Pick a metric ρ on Γ, say the word-metric:

$$\boxed{d_n(\omega) = \rho(Z_n(\omega))}$$



$\omega = (\omega_0, \omega_1, \ldots)$

$Z_n(\omega) = \omega_0 \cdots \omega_{n-1}$

paths in $\mathrm{Cay}(\Gamma, S)$

$$\ell_n := \int_\Omega d_n(\omega)\, d\mathbb{P}_\mu(\omega) = \int_{G^n} \rho(\omega_0\omega_1\cdots\omega_{n-1})\, d\mu(\omega_0)\cdots d\mu(\omega_{n-1})$$

$$= \int_G \rho(g)\, d\mu^{*n}(g)\ ,\ n \geq 1.$$

$\ell_{n+m} \leq \ell_n + \ell_m \quad \forall n, m \geq 0 \quad \Delta$-ineq.

$\ell := \lim_n \frac{\ell_n}{n} = \inf_n \frac{\ell_n}{n} \geq 0.$ (Drift of ρ)

Rmk. ℓ is very difficult to calculate.

Basic facts: $\Gamma = \mathbb{Z}^d$ — $\boxed{\ell_S = 0}$

$\forall S \subset \mathbb{Z}^d$ (all metrics)

$|S| < +\infty$, $S = S^{-1}$.

Γ = nilpotent groups.

$\Gamma = \mathbb{F}_d$, $d \geq 2$, then $\ell_S > 0$ for every $S \subset \mathbb{F}_d$, $|S| < +\infty$.

(In fact: Same holds for any non-amenable group)

Thm (Furstenberg - Kesten, Kingman)

$$\ell = \lim_{n \to \infty} \frac{d_n(\omega)}{n} \quad \text{a.e. } \omega \ [\mathbb{P}_\mu].$$

Refined questions:

What can we say about

$$\frac{d_n(\omega) - n\ell}{\cdots} \Rightarrow ?$$

Thm (Sawyer-Steger) $\Gamma = \mathbb{F}_d$, $d \geq 2$

$$\frac{d_n(\omega) - n\ell}{\sqrt{n}} \xrightarrow{d} N(0,\sigma), \quad \sigma > 0.$$

$\Longleftrightarrow \mu^{*n}\left(g : \rho(g) \leq \sqrt{n}\, t + n\ell\right) \longrightarrow \int_{-\infty}^{t} \frac{e^{-x^2/2\sigma}\,dx}{\sqrt{2\pi\sigma}}$

for some $\sigma > 0$.

<u>Thm</u> (B) The same is true for all N.E. Gromov hyperbolic groups and also direct products of these (for any generating set)

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<u>Harmonic fcns on groups</u>

μ = prob. measure on Γ.

<u>Def</u>. $f : \Gamma \to \mathbb{R}$ <u>μ-integrable</u> is <u>left-μ-harmonic</u> if

$$\forall g \quad \mu * f(g) = \int f(kg)\, d\mu(k) = f(g)$$

$$\mu = \frac{1}{|S|} \sum_{s \in S} \delta_s$$

$f : \Gamma \to \mathbb{R}$ is <u>right-μ-harmonic</u>

$$\int f(gk)\, d\mu(k) = f(g) \quad \forall g.$$

$f : \Gamma \to \mathbb{R}$ is <u>bi-μ-harmonic</u> if both left and right.

Fact: $\nexists$ non-trivial, bnd. bi-harmonic fcns.
(Kaimanovich)
On the other hand - plenty of unbnd. ones.

If $f: \Gamma \to \mathbb{R}$ is bi-μ-harmonic and $\sup_x |f(xg) - f(x)| \le C_g < +\infty \ \forall g$, then $\frac{f(Z_n(\omega))}{\sqrt{n}} \xrightarrow{d} N(0,\sigma)$, $\sigma \geq 0$.
Conseq. of MCLT.

Goal: Approximate $\rho(Z_n(\omega)) - n\ell$ by $f(Z_n(\omega))$ for some ~~bnd~~ bi-harmonic fcn.

Def. $f: \Gamma \to \mathbb{R}$ is μ-quasi-harmonic if $\mu * f(g) = f(g) + \ell$

If f is left-μ-quasiharmonic

$$\left.\begin{array}{l}\tilde{f}(g,n) = f(g) - n\ell \\ \tilde{\mu} = \mu \times \delta_1 \text{ on } \Gamma \times \mathbb{Z}\end{array}\right\} \tilde{\mu} * \tilde{f} = \tilde{f}$$

Main ideas:

1) $\rho(g) \geq f(g)$, $\mu * f = f + \ell$

ℓ = drift of ρ.

2) Every left-μ-quasi-harm. fcn on Γ is on an "almost bnd" distance away from a bi-μ-quasi-harm. fcn.

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$$\frac{\rho(Z_n(\omega)) - n\ell}{\sqrt{n}}$$

$$\approx \frac{f(Z_n(\omega)) - n\ell}{\sqrt{n}}$$