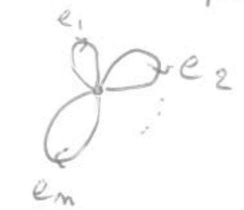
1 Outer space

$X_n =$ 

$F_n = \pi_1 X_n = \langle x_1, \dots, x_n \rangle$

Want to study $\mathrm{Aut}\, F_n$, $\mathrm{Out}\, F_n = \mathrm{Aut}\, F_n / \mathrm{Inn}\, F_n$ via action on a space.

$\rightsquigarrow$ Culler-Vogtmann's Outer space cv_n

A point in cv_n is

- Γ finite metric graph (all vertices of degree ≥ 3)
- $X_n \xrightarrow[\sim]{\varphi \text{ homotopy equivalence}} \Gamma$ "marking"

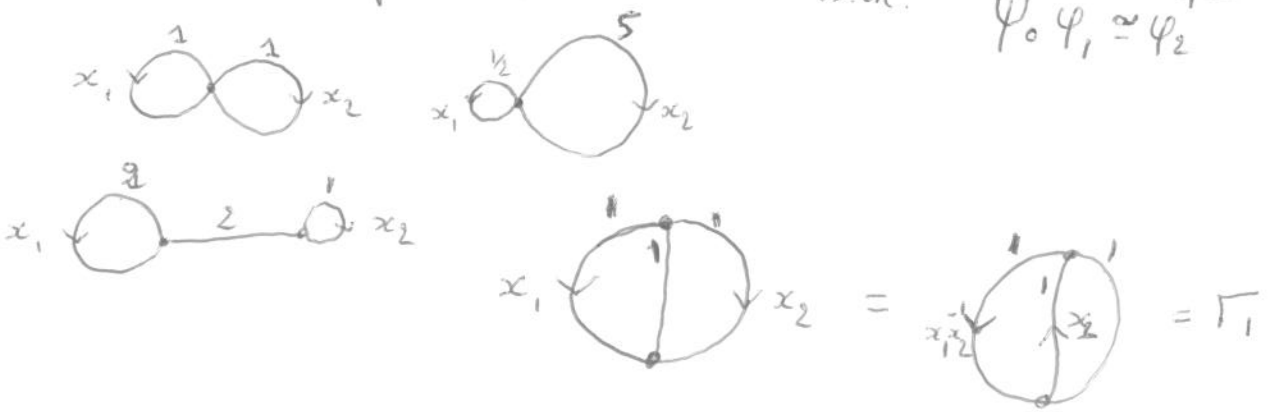
modulo $\sim$:

$(\Gamma_1, \varphi_1) \sim (\Gamma_2, \varphi_2)$ if

$$\begin{tikzcd} & \Gamma_1 \arrow[d, "\exists \text{ isometry } \psi"] \\ X_n \arrow[ur, "\varphi_1"] \arrow[r, "\varphi_2"'] & \Gamma_2 \end{tikzcd}$$

s.th. $\psi \circ \varphi_1 \overset{\text{homotopic}}{\simeq} \varphi_2$

Combinatorial picture:



Lengths: $\left.\begin{array}{l} g \in F_n = \pi_1 X_n \\ (\Gamma, \varphi) \in cv_n \end{array}\right\} \to \|g\|_\Gamma = \Gamma$-length of shortest circuit freely homotopic to $\varphi(g)$

$\|x_1\|_{\Gamma_1} = 2$ $\quad \|x_1 x_2^{-1}\|_{\Gamma_1} = 2$

$\|x_1 x_2\|_{\Gamma_1} = 4$

$\mathrm{Aut}\, F_n$ acts on CV_n : $\alpha \in \mathrm{Aut}\, F_n \rightsquigarrow \tilde{\alpha} : X_n \to X_n$ H.E.

$\Rightarrow$ $(\Gamma, \varphi) \cdot \alpha = (\Gamma, \varphi \circ \tilde{\alpha})$

$\mathrm{Inn}\, F_n$ acts trivially $\Rightarrow$ $\mathrm{Out}\, F_n$ acts on CV_n.

- Topology on CV_n ?

$\| \cdot \|_\Gamma : F_n \to \mathbb{R}^+$ yields a map $CV_n \to \mathbb{R}^{F_n}$

$\Gamma \mapsto \| \cdot \|_\Gamma$

Theorem (Culler - Morgan) This map is injective:
- The associated length function determines a point in Outer Space
- Lengths of elements parametrize points in Outer Space.

$\rightsquigarrow$ induced topology

Theorem (Culler - Vogtmann) CV_n is contractible.

2. Spectral rigidity

Definition: $X \subseteq F_n$ is spectrally rigid if
$\|x\|_{\Gamma_1} = \|x\|_{\Gamma_2} \ \forall x \in X \Rightarrow \Gamma_1 = \Gamma_2$

Theorem (Smillie - Vogtmann) no finite subset of F_n is spectrally rigid.

Goal: find sparse spectrally rigid subsets of F_n

Origin of Outer Space:

Σ_g closed surface of genus g

Teichmüller Space $= \left\{ \Sigma_g \xrightarrow{\varphi \text{ homeo}} (\Sigma, \underset{\text{hyperbolic metric}}{\rho}) \right\} / \sim$

$$MCG(\Sigma_g) = \mathrm{Homeo}(\Sigma_g) / \mathrm{Homeo}_0(\Sigma_g) \underset{\substack{\uparrow \\ \text{Dehn-Nielsen} \\ \text{Baer}}}{\cong} \mathrm{Out}(\pi_1 \Sigma_g)$$

$MCG(\Sigma_g)$ acts on Teichmüller space.

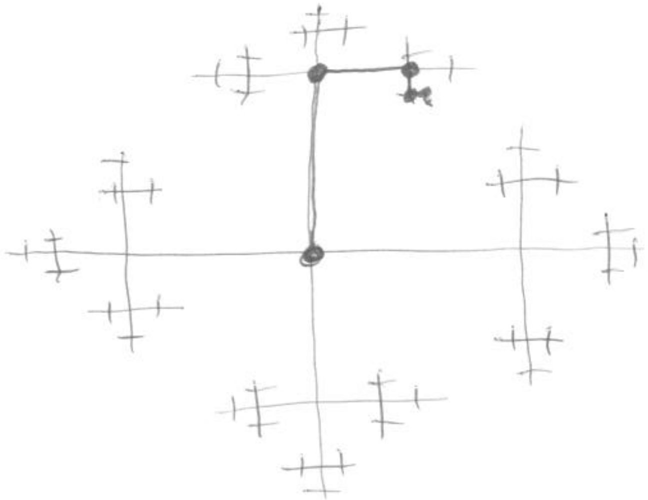
$\rightsquigarrow$ Length function: $\| \cdot \|_{(\Sigma, \rho)} : \pi_1 \Sigma_g \to \mathbb{R}$

<u>Fact</u>: There is a <u>finite</u> set of curves $\gamma_1, \dots, \gamma_k$ on Σ_g such that any point (Σ, ρ) in Teichmüller Space is determined by

$$\left(\|\gamma_1\|_{(\Sigma,\rho)}, \dots, \|\gamma_k\|_{(\Sigma,\rho)} \right)$$

$\{\gamma_1, \dots, \gamma_k\}$ is called <u>spectrally rigid</u>.

<u>Theorem</u> (I. Kapovich) The trace of almost every non-backtracking random walk on $Cay(F_n, \{x_1, \dots, x_n\})$ is spectrally rigid.



<u>Theorem</u> (C. - Francaviglia - Kapovich - Martino)

$(n \geqslant 3)$. $H \trianglelefteq Aut(F_n)$ with infinite image in $Out(F_n)$,

$$g \underset{\neq 1}{\in} F_n$$

$\Rightarrow$ The orbit Hg is spectrally rigid.

In particular, $g = x_1$, $H = Aut(F_n)$

$Hg = Aut(F_n) \cdot x_1$ = set of primitive elements is spectrally rigid.

<u>Theorem</u> (Ray) $H = \langle \alpha \rangle$ cyclic

$$g \in F_n$$

$\Rightarrow Hg = \{ \alpha^k(g) \mid k \in \mathbb{Z} \}$ is not spectrally rigid

<u>Thm</u> (Duchin, Leininger, Rafi)

A set X of simple closed curves in Σ_g is spectrally rigid w.r.t. the space of flat metrics on Σ_g $\iff \overline{X} = \mathcal{PMF}$

In particular, finite sets are never spectrally rigid.

3. Lipschitz distorsion

$$\begin{tikzcd} & \Gamma_1 \\ X_n \arrow[ur, "\varphi_1"] \arrow[dr, "\varphi_2"'] & \\ & \Gamma_2 \end{tikzcd} \qquad D(\Gamma_1, \Gamma_2) = \sup_{\substack{g \in F_n \\ g \neq 1}} \frac{\|g\|_{\Gamma_2}}{\|g\|_{\Gamma_1}} < +\infty$$

<u>Theorem</u> (T. White)

$$D(\Gamma_1, \Gamma_2) = \inf \left\{ K \;\middle|\; \begin{array}{l} \exists\, \psi : \Gamma_1 \to \Gamma_2 \;\; K\text{-Lipschitz} \\ \psi \simeq \varphi_2 \circ \varphi_1^{-1} \end{array} \right\}$$

and the inf is realized by a map ψ.

If $\mathrm{vol}(\Gamma_1) = \mathrm{vol}(\Gamma_2) = 1 \Rightarrow d(\Gamma_1, \Gamma_2) = \log D(\Gamma_1, \Gamma_2)$

$\rightsquigarrow$ d satisfies $\triangle$ inequality

$d(\Gamma_1, \Gamma_2) = 0 \iff \Gamma_1 = \Gamma_2$

However $D(\Gamma_1, \Gamma_2)$ need not be $= D(\Gamma_2, \Gamma_1)$!

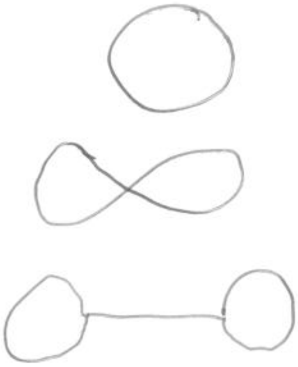
- Bestvina used D to show the existence of train tracks for fully irreducible automorphisms of F_n.

$$D(\Gamma_1, \Gamma_2) = \sup_{\substack{g \in F_n \\ g \neq 1}} \frac{\|g\|_{\Gamma_2}}{\|g\|_{\Gamma_1}}$$

<u>Prop</u>: Given $\Gamma_1 \in cv_n$, $\exists$ finite set of primitive elements $\{g_1, \dots, g_k\} \subseteq F_n$ such that $\forall \Gamma_2 \in cv_n$

$$D(\Gamma_1, \Gamma_2) = \max_{1 \leq i \leq k} \frac{\|g_i\|_{\Gamma_2}}{\|g_i\|_{\Gamma_1}}$$

One can take the g_i's to be elements corresponding to "almost simple curves" in Γ_1



<u>Corollary</u>: The set of primitive elements is spectrally rigid

Suppose $\|g\|_{\Gamma_1} = \|g\|_{\Gamma_2}$ $\forall g$ primitive

$$\Rightarrow \forall h \in F_n,\ h \neq 1, \quad \frac{\|h\|_{\Gamma_2}}{\|h\|_{\Gamma_1}} \leq D(\Gamma_1, \Gamma_2) \overset{\text{Prop}}{=} \max \frac{\|g_i\|_{\Gamma_2}}{\|g_i\|_{\Gamma_1}} = 1$$

similarly $\quad \dfrac{\|h\|_{\Gamma_1}}{\|h\|_{\Gamma_2}} \leq 1$

$$\Rightarrow \|h\|_{\Gamma_1} = \|h\|_{\Gamma_2} \quad \forall h \in F_n \quad \Rightarrow \quad \Gamma_1 = \Gamma_2 .$$

4 Currents on free groups.

* g not primitive $\rightsquigarrow$ Aut $F_n \cdot g$ spectr. rigid?

Idea: approximate primitive elements.

Ex: $F_3 = \langle a, b, c \rangle$, $g = a b a^{-1} c$

$$\alpha \in \operatorname{Aut} F_n : \quad \alpha(a) = ab, \quad \alpha(b) = b, \quad \alpha(c) = c \quad \Rightarrow \quad \alpha(g) = g, \quad \alpha^n(g) = g.$$

$$\beta \in \operatorname{Aut} F_n : \quad \beta(b) = bc, \quad \beta(a) = a, \quad \beta(c) = c \quad \Rightarrow \quad \beta(g) = a\,b\,c\,a^{-1}c, \quad \beta^n(g) = a\,b\,c^n a^{-1} c$$

$$\boxed{\|c\|_\Gamma = \lim_{n\to\infty} \frac{\|\beta^n(g)\|_\Gamma}{n}} \quad \text{for any } \Gamma \in cv_n$$

$\rightsquigarrow$ can recover lengths of primitive elements.

$\rightsquigarrow$ Aut $F_n \cdot g$ is spectrally rigid.

Curr(F_n) is a top. space naturally containing the conjugacy classes of F_n $\quad g \in F_n \rightsquigarrow \eta_g \in \mathrm{Curr}(F_n)$

- $\mathbb{R}^+$ - linear structure
- $\mathrm{Out}(F_n) \curvearrowright \mathrm{Curr}(F_n)$
- Span$\{\eta_g \mid g \in F_n\}$ is dense.

<u>Prop:</u> There is an "intersection form" $\langle \cdot , \cdot \rangle$

$$cv_n \times \mathrm{Curr}(F_n) \to \mathbb{R}^+$$

with
- continuous, $\mathbb{R}^+$-linear in 2^{nd} component.
- $\langle \Gamma, \eta_g \rangle = \|g\|_\Gamma$
- $\mathrm{Out}(F_n)$-~~equivariant~~ invariant

- $\mathbb{P}\mathrm{Curr}(F_n)$ is compact.

<u>Prop:</u> $\mathcal{M}_n = \overline{\{[\eta_g] \mid g \text{ is primitive}\}} \subseteq \mathbb{P}\mathrm{Curr}(F_n)$ is the smallest nonempty closed $\mathrm{Out}(F_n)$ invariant subset. $(n \geq 3)$

<u>Idea of Pf of thm</u> $\quad H \trianglelefteq \mathrm{Aut}(F_n)$

$g \neq 1$

Handel-Mosher $\Rightarrow \exists$ fully irreducible $\alpha \in H$

Prove that $\overline{\{[\eta_h] \mid h \in H g\}} = Z$ contains a nonempty closed $\mathrm{Out}(F_n)$-invar. subset.