Introduction to Artin Groups

Lectures 1 + 2:

Coxeter + Artin Groups

Lectures 3 + 4:

Right-angled Artin Groups

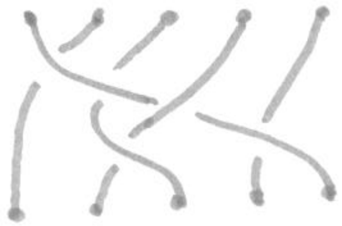
Coxeter and Artin Groups

Warm-up: braid groups

B_n = n-strand braid group

Several descriptions:

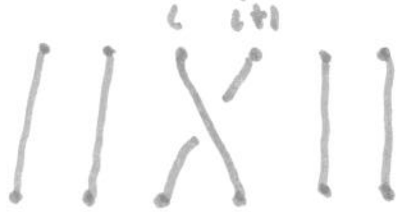
- As braided strings



(up to isotopy)

multiplication = concatenation

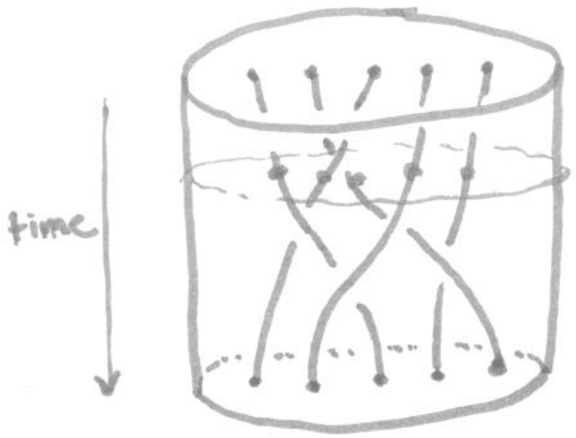
- By a presentation

generators: $s_i =$ 

$$B_n = \left\langle s_1, \dots, s_{n-1} \;\middle|\; \begin{array}{l} s_i s_j = s_j s_i \text{ if } |i-j|>1 \\ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \end{array} \right\rangle$$

 = 

- Topological description



Think of a braid as n points moving around in the complex plane.

This gives a path in the configuration space

$$\mathcal{C}_n = \{ (z_1, \dots, z_n) \in \mathbb{C}^n \mid z_i \neq z_j \ \forall i,j \}$$

Σ_n = symmetric group acts freely on $\mathcal{C}_n$

braid $\rightsquigarrow$ loop in $\mathcal{C}_n / \Sigma_n$

<u>Thm</u> (Hurwitz Fox - Neuwirth '62)

$$B_n \cong \pi_1 (\mathcal{C}_n / \Sigma_n)$$

Also note that

$$\mathcal{C}_n = \mathbb{C}^n - \bigcup_{i,j} H_{ij}$$

$H_{ij} = \{ (z_1, \dots z_n) \mid z_i = z_j \}$ hyperplane

= fixed set of reflection $\tau_{ij} \curvearrowright \mathbb{C}^n$

$$B_n \cong \pi_1 (\text{hyperplane complement} / \Sigma_n)$$

These are classical examples of

Coxeter group Σ_n

Artin group B_n

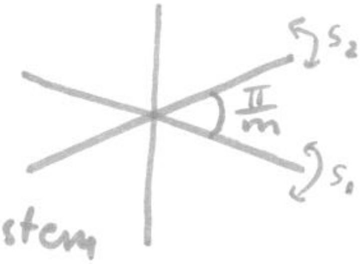
Definitions

$\boxed{\text{Coxeter system}}$ (W, S)

$S = \{s_1, \dots, s_n\}$

$W = \langle S \mid s_i^2 = 1,\ (s_i s_j)^{m_{ij}} = 1 \rangle \quad m_{ij} \in \{2, 3, \dots, \infty\}$

$= \text{reflection gp} \curvearrowright \mathbb{R}^n$



Associated to every Coxeter system is a $\boxed{\text{Artin system}}$ (A, S)

$A = \langle S \mid \underbrace{s_i s_j s_i \cdots}_{m_{ij}} = \underbrace{s_j s_i s_j \cdots}_{m_{ij}} \rangle$

Note: if $s_i = s_i^{-1}$, $(s_i s_j)^{m_{ij}} = 1 \Longleftrightarrow s_i s_j s_i \cdots = s_j s_i s_j \cdots$

So $A \twoheadrightarrow W$.

Remark on terminology:

Artin introduced braid groups

Tits developed theory of Artin gps

Should be called Artin-Tits gps!

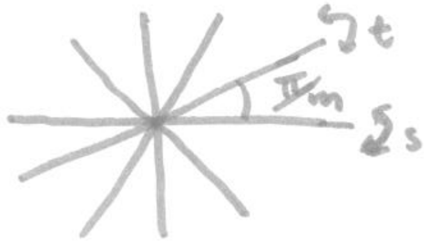
Two types of Artin groups

① $\boxed{\text{Spherical type}}$

A spherical $\iff W$ finite

$\iff W \curvearrowright S^{n-1}$

Eg. 1) $W = D_{2m} = \langle s, t \mid s^2 = t^2 = (st)^m = 1 \rangle$



$A = \langle s, t \mid stst\ldots = tsts\ldots \rangle$

2) $W = \Sigma_n$, $A = B_n$

② $\boxed{\text{Non-spherical type}}$

A non-spherical $\iff W$ infinite

Eg. 1) Some $m_{ij} = \infty \Rightarrow (s_i s_j)$ has ∞ order

$W = D_\infty = \langle s, t \mid s^2 = t^2 = 1 \rangle$

$A = \langle s, t \rangle = F_2$

2) W = affine reflection gp

<u>Geometric connection</u> between
Coxeter gp W and Artin gp A

$W \circlearrowright \mathbb{R}^n$ as reflection gp

complexify action

$W \circlearrowright \mathbb{C}^n$

reflection hyperplanes

$H_r = \{\text{fixed set of } r\}$ $r \in W$ reflection

Let
$$\mathcal{C}_W = \mathbb{C}^n - \bigcup_r H_r$$

Then W acts freely on $\mathcal{C}_W$ and

<u>Thm</u> (Brieskorn '71, van der Lek '83)

$\pi_1(\mathcal{C}_W/W) \cong A$, associated Artin gp

<u>Thm</u> (Deligne '72) If W is <u>finite</u> ($\Leftrightarrow$ A spherical type) then $\mathcal{C}_W/W$ is aspherical, hence it is a $K(A,1)$-space. ($\widetilde{\mathcal{C}_W}$ is contractible)

For non-spherical A, this is an open question.

$\boxed{\text{$K(\pi,1)$-conjecture: For any A, $\mathcal{C}_W/W$ is a $K(A,1)$-space.}}$

Some partial results:

Notation: (A,S) Artin system

For $T \subseteq S$

$$A_T = \text{subgp of } A \text{ generated by } T$$
$$\cong \text{Artin system } (A_T, T)$$

call these parabolic subgroups.

Say A_T is 2-spherical if

$$s_i, s_j \in T \Rightarrow m_{ij} \neq \infty$$

Note: in general, 2-spherical $\not\Rightarrow$ spherica

Eg: affine Artin gps

Thm (C-Davis '95) Suppose (A,S) satisfie

$(*)$ A_T 2-spherical $\Rightarrow$ A_T spherical

Then the $K(\pi,1)$-conjecture holds for A

Thm (Ellis-Skoldberg, Godelle-Paris)

If the $K(\pi,1)$-conj holds for every 2-spherical parabolic subgp of A, then it holds for A.

A crash course in CAT(0) geometry

(X, d) geodesic metric space

X is CAT(0) if triangles in X are at least as thin as their comparison triangles in $\mathbb{E}^2$.

X CAT(0) $\Longrightarrow$

- X has unique geodesics

- geodesics vary continuously with endpoints

X CAT(0) $\Longrightarrow$ X is contractible

In general, it is very difficult to prove a metric space is CAT(0). Two powerful theorems:

Thm: Simply conn + locally CAT(0) $\Longrightarrow$ CAT(0).

Thm: If X has a cubical metric then

X locally CAT(0) $\Longleftrightarrow$ $\underbrace{\text{links of vertices are flag}}_{\text{purely combinatorial}}$

Idea of proof:

Proof in spherical case (Deligne):

- Construct simplicial complex
  $D_W \simeq \widetilde{\mathcal{C}}_W$ = univ cover of $\mathcal{C}_W/W$
- Prove D_W contractible

Proof in (*) case (C-Davis):

- Construct simplicial complex
  $D_W \simeq \widehat{\mathcal{C}}_W$
- Give D_W a cubical metric
- Prove D_W is CAT(0) $\Longleftrightarrow$ ~~when~~ (*) holds

<u>Conj</u>: D_W always supports a CAT(0) metric (the Moussong metric)

$$\Downarrow$$

$K(\pi,1)$ - Conj

<u>Thm</u> (Salvetti, C-Davis) For any (A,S), $\exists$ a finite cell complex $\mathrm{Sal}(A)$ homotopy equiv to $\mathcal{C}_W/W$.

In particular

$$K(\pi,1)\text{-conj} \iff \mathrm{Sal}(A) \text{ is a } K(A,1)\text{-space}$$

<u>Cor</u> If $K(\pi,1)$-conj holds for A, then A has a finite $K(A,1)$-space, hence

- A is torsion-free
- $cd(A) \leq \dim(\mathrm{Sal}(A)) = \max\{|T| \mid A_T \text{ spherical}$
- can compute $H^*(A)$

$\vdots$

Sal(A) : has one cell C_T for each spherical parabolic $A_T \subset A$.

C_T = Coxeter cell for W_T (W_T finite)

$C_T =$ 

$W_T = \langle s, t \mid s^2 = t^2 = (st)^3 =$

Each face of C_T is the Coxeter cell $C_{T'}$ for some $T' \subset T$.

$$Sal(A) = \coprod_{W_T \text{ finite}} C_T \Big/ \text{glue corresponding faces}$$

Eg: $A = \langle s, t \mid sts = tst \rangle$

$Sal(A) = C_\emptyset \cup C_{\{s\}} \cup C_{\{t\}} \cup C_{\{s,t\}}$

$=$ 

$\widetilde{Sal}(A)$ = Cayley 2-complex for A

Algorithmic Properties

Fundamental problems (Max Dehn):

Word Problem: Given a presentation $G = \langle S \mid R \rangle$, decide whether a word w in S represents the trivial elt in G.

Conjugacy Problem: Given $G = \langle S \mid R \rangle$, decide whether two words w_1, w_2 represent conjugate elts in G.

If such algorithms exist, what is their complexity?

Gromov: A δ-hyperbolic group has a presentation with a linear time solution to the word problem and solvable conj problem.

Biautomatic groups have quadratic time solutions to the word problem and solvable conjugacy problem.

<u>Thm</u> (C): If A is a spherical Artin group then A has nice normal forms which give rise to a biautomatic structure.

Garside: normal forms for braids

Brieskorn - Saito: normal forms for other spherical A.

C - biautomatic structure

Dehornoy - Paris: theory of "Garside Groups"

Using these normal forms one can show (A spherical)

- center $(A) \cong \mathbb{Z}$
- A is torsion-free
- A is linear (Bigelow, Krammer, Cohen-Wales, Digne)

$\vdots$

Summary

If A is <u>spherical</u>, the following hold

- nice, finite $K(A,1)$-spaces
- biautomatic
- torsion-free
- Center$(A) \cong \mathbb{Z}$
- linear

If A is <u>non-spherical</u> none of the above are known except in special cases!

Open question for spherical A:
Is A a CAT(0) group?
(Does A act properly, cocompactly on a CAT(0) space?)

Note: Except for A = free gp,
<u>no</u> Artin gp is δ-hyperbolic.

$A \neq$ free gp $\Rightarrow$ some $m_{ij} \neq \infty$
$\Rightarrow A_T$ spherical for $T = \{s_i, s_j\}$
$\Rightarrow$ Center$(A_T) \cong \mathbb{Z}$
$\Rightarrow \mathbb{Z}^2 \subset A_T \subset A$

Some partial results on CAT(0) question:
Bell, Brady-Crisp, T. Brady, Bestvina